MAC2312 Section U03
Suggested problems for Test 1
(Test 1 is Friday February 10th, in class)

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1. Write out the terms of the sum, and evaluate.

$$\sum_{k=1}^{3} k^3 .$$

$1^3 + 2^3 + 3^3 = 1 + 8 + 27$

$= 36$

2. Write out the terms of the sum, and evaluate.

$$\sum_{j=2}^{6} (3j-1)$$

$(3\cdot 2 - 1) + (3\cdot 3 - 1) + (3\cdot 4 - 1) + (3\cdot 5 - 1) + (3\cdot 6 - 1)$

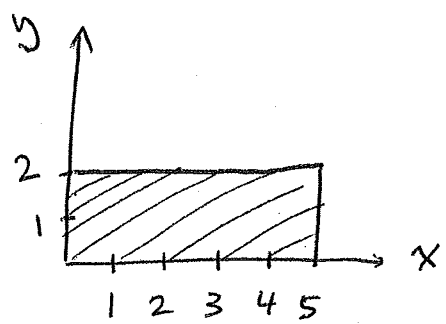
$= (6-1) + (9-1) + (12-1) + (15-1) + (18-1)$

$= 5 + 8 + 11 + 14 + 17$

$= 55$ (Shortcut: Five equally spaced numbers whose average value is 11)

3. Evaluate the integral. You can use geometry if it helps you.

$$\int_0^5 2\,dx$$



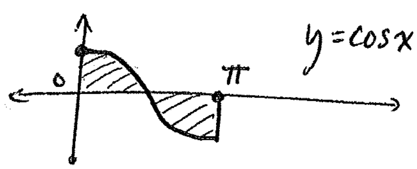
$$2 \times 5 = 10$$

4. Evaluate the integral. You can use geometry if it helps you.

$$\int_0^{\pi} \cos x\,dx$$

$$\left[\sin x\right]_0^{\pi} = \sin\pi - \sin 0 = 0 - 0 = 0$$

Also, look at graph:



Negative height perfectly balances positive height

5. Evaluate the integral. You can use geometry if it helps you.

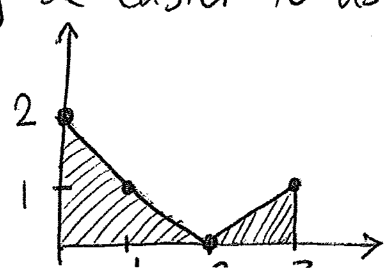
$$\int_0^3 |x-2|\,dx$$

$$\int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx$$

$$= \int_0^2 (2-x)\,dx + \int_2^3 (x-2)\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$$

May be easier to use graph:



Total area $= 2.5$ or $\frac{5}{2}$

6. Evaluate the integral.

$$\int_1^4 \frac{4}{x^2}\,dx = \int_1^4 4x^{-2}\,dx$$

$$= 4\left[\frac{x^{-1}}{-1}\right]_1^4 = 4\left[-\frac{1}{x}\right]_1^4 = 4\left[\frac{1}{x}\right]_4^1$$

$$= 4\left(1-\frac{1}{4}\right) = 4\left(\frac{3}{4}\right) = 3$$

7. Evaluate the integral.

$$\int_1^2 \frac{1}{x^6}\,dx = \int_1^2 x^{-6}\,dx$$

$$= \left[\frac{x^{-5}}{-5}\right]_1^2 = \left[-\frac{1}{5x^5}\right]_1^2 = \left[\frac{1}{5x^5}\right]_2^1$$

$$= \frac{1}{5}\left[\frac{1}{x^5}\right]_2^1 = \frac{1}{5}\left(1-\frac{1}{32}\right) = \frac{1}{5}\cdot\frac{31}{32} = \frac{31}{160}$$

8. Evaluate the integral.

$$\int_4^9 2x\sqrt{x}\,dx$$

$$\int_4^9 2x^{3/2}\,dx = \left[2\,\frac{x^{5/2}}{5/2}\right]_4^9 = \frac{4}{5}\left[x^{5/2}\right]_4^9$$

$$= \frac{4}{5}\left(9^{5/2}-4^{5/2}\right) = \frac{4}{5}\left(3^5-2^5\right)$$ Realistically, could leave it like that. 3^5 is a big number.

9. Evaluate the integral.

$$\int_0^{\pi/4} \sec^2\theta\,d\theta$$

$$\left[\tan\theta\right]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0$$

$$= 1 - 0 = 1$$

10. Evaluate the integral.

$$\int_{\ln 2}^{3} 5e^x \, dx$$

$$\left[5e^x\right]_{\ln 2}^{3} = 5\left[e^x\right]_{\ln 2}^{3}$$

$$= 5\left(e^3 - e^{\ln 2}\right) = 5\left(e^3 - 2\right)$$

11. Evaluate the integral.

$$\int_{1/2}^{1} \frac{1}{2x} \, dx = \frac{1}{2}\int_{1/2}^{1} \frac{1}{x} \, dx$$

$$= \frac{1}{2}\left[\ln|x|\right]_{1/2}^{1} = \frac{1}{2}\left(\underbrace{\ln 1}_{=0} - \underbrace{\ln \tfrac{1}{2}}_{=-\ln 2}\right)$$

$$= \frac{1}{2}\left(-(-\ln 2)\right) = \frac{\ln 2}{2}$$

12. Evaluate the integral.

$$\int_{1}^{4} \frac{|2-x|}{x} \, dx$$

$$\int_{1}^{2} \frac{|2-x|}{x} \, dx + \int_{2}^{4} \frac{|2-x|}{x} \, dx$$

$$= \int_{1}^{2} \frac{2-x}{x} \, dx + \int_{2}^{4} \frac{x-2}{x} \, dx$$

$$= \int_{1}^{2} \left(\frac{2}{x} - 1\right) dx + \int_{2}^{4} \left(1 - \frac{2}{x}\right) dx$$

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$$= \left[2\ln|x| - x\right]_{1}^{2} + \left[x - 2\ln|x|\right]_{2}^{4}$$

$$= (2\ln 2 - 2) - (1\ln 1 - 1) + (4 - 2\ln 4) - (2 - 2\ln 2)$$

$$= 2\ln 2 - 2 + 1 + 4 - 2\ln 4 - 2 + 2\ln 2 = \ldots = 1$$

13. Find the position function of a particle, given the following.

$a(t) = t^2 - 3t + 1$ (acceleration function)
$v(0) = 0$ (velocity at time 0)
$s(0) = 0$ (position at time 0)

$$v(t) = \int a(t)\,dt = \int (t^2 - 3t + 1)\,dt = \frac{t^3}{3} - 3\frac{t^2}{2} + t + \underset{\substack{\uparrow \\ V(0) \\ \text{or } V_0}}{C}$$

$$v(t) = \frac{t^3}{3} - \frac{3}{2}t^2 + t$$

$$s(t) = \int v(t)\,dt = \int \left(\frac{t^3}{3} - \frac{3}{2}t^2 + t\right)dt = \frac{t^4}{12} - \frac{3}{2}\frac{t^3}{3} + \frac{t^2}{2} + \underset{\substack{\uparrow \\ s(0) \\ \text{or } s_0}}{C}$$

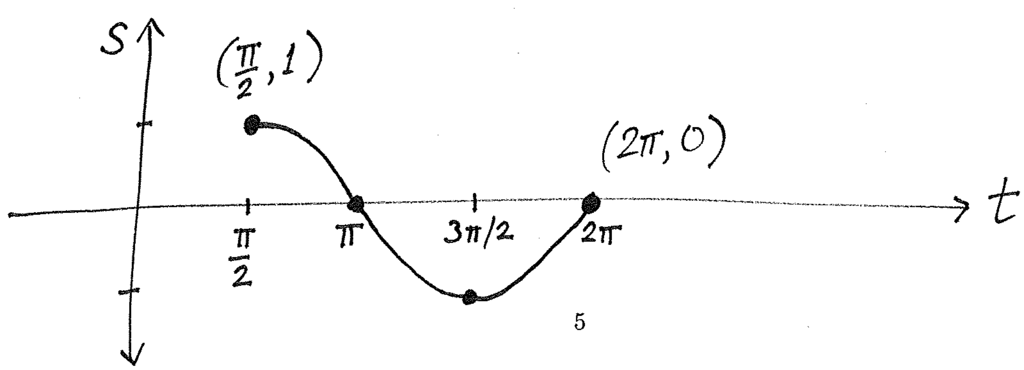
$$s(t) = \frac{t^4}{12} - \frac{t^3}{2} + \frac{t^2}{2}$$

14. A particle moves with a velocity $v(t) = \cos t$ (in meters per second) and starts at position 0. Find the position function of the particle, and draw a graph of the position function over the time interval $\pi/2 \le t \le 2\pi$.

$$s(t) = \int v(t)\,dt = \int \cos t\,dt = \sin t + C$$

$s(0) = 0$ but also $s(0) = \sin 0 + C = 0 + C = C$.

We conclude $C = 0$, so $s(t) = \sin t + 0 = \underline{\sin t}$



15. Find the average value of the function $f(x) = 3x$ over the interval $[1, 3]$.

$$\frac{1}{3-1}\int_1^3 3x\,dx = \frac{1}{2}\left[\frac{3x^2}{2}\right]_1^3$$

$$= \frac{3}{4}\left[x^2\right]_1^3 = \frac{3}{4}(9-1) = \frac{3}{4}\cdot 8 = 6$$

16. Find the average value of the function $f(x) = \sin x$ over the interval $[0, \pi]$.

$$\frac{1}{\pi-0}\int_0^\pi \sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\cos x\right]_\pi^0 = \frac{1}{\pi}\left(\cos 0 - \cos \pi\right) = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}$$

17. Find the average value of the function $f(x) = \frac{1}{x}$ over the interval $[1, e]$.

$$\frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\left[\ln|x|\right]_1^e$$

$$= \frac{1}{e-1}\left(\ln e - \ln 1\right)$$

$$= \frac{1}{e-1}\left(1 - 0\right) = \frac{1}{e-1}$$

18. Evaluate the integral.

$$\int_0^1 (2x+1)^3\,dx$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=3$

$$\int_1^3 u^3 \cdot \frac{1}{2}du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^3 = \frac{1}{8}\left[u^4\right]_1^3 = \frac{1}{8}(3^4-1)$$

$$= \frac{1}{8}(81-1)$$

$$= 10$$

19. Evaluate the integral.

$$\int_0^8 x\sqrt{1+x}\,dx$$

$u-1=x$

$u=1+x$

$du = 1dx$

$x=0 \Rightarrow u=1$

$x=8 \Rightarrow u=9$

$$\int_1^9 (u-1)u^{1/2}\,du = \int_1^9 \left(u^{3/2}-u^{1/2}\right)du$$

$$= \left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right]_1^9$$

$$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{2}{5}\left[u^{5/2}\right]_1^9 - \frac{2}{3}\left[u^{3/2}\right]_1^9$$

$$= \frac{2}{5}(3^5-1) - \frac{2}{3}(3^3-1)$$

20. Evaluate the integral.

$$\int_0^{\pi/6} \cos 3x\,dx$$

$u = 3x$

$du = 3dx$

$\frac{1}{3}du = dx$

$x=0 \Rightarrow u=0$

$x=\frac{\pi}{6} \Rightarrow u=\frac{\pi}{2}$

$$\int_0^{\pi/2} \cos u \cdot \frac{1}{3}du$$

$$= \frac{1}{3}\left[\sin u\right]_0^{\pi/2}$$

$$= \frac{1}{3}(1-0) = \frac{1}{3}$$

21. Evaluate the integral.

$$\int_1^2 \sqrt{5x-1}\,dx$$

$u = 5x - 1$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$x = 1 \Rightarrow u = 4$

$x = 2 \Rightarrow u = 9$

$$\int_4^9 \sqrt{u}\cdot\frac{1}{5}\,du = \frac{1}{5}\int_4^9 u^{1/2}\,du = \frac{1}{5}\left[\frac{u^{3/2}}{3/2}\right]_4^9$$

$$= \frac{1}{5}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^9 = \frac{2}{15}\left(3^3 - 2^3\right) = \frac{38}{15}$$

22. Evaluate the integral.

$$\int_0^{\pi/4} \sin x \cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$x = 0 \Rightarrow u = 0$

$x = \pi/4 \Rightarrow u = 1/\sqrt{2}$

$$\int_0^{1/\sqrt{2}} u\,du = \left[\frac{u^2}{2}\right]_0^{1/\sqrt{2}} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

23. Find the area bounded by the curves $y = x^2$ and $y = 2x$.

Intersection points? $x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0 \quad x = 0,\ x = 2$

Test $x = 1$: $2x$ has bigger outputs than x^2

Area: $\int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$

24. Find the area bounded by the curves $y = x^2$ and $y = x + 2$.

Intersection points? $x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \quad x = -1,\ x = 2$

Test $x = 0$: $x + 2$ has bigger outputs than x^2

Area: $\int_{-1}^2 (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2$

$= \frac{1}{2}\left[x^2\right]_{-1}^2 + 2\left[x\right]_{-1}^2 - \frac{1}{3}\left[x^3\right]_{-1}^2$

$= \frac{1}{2}(4-1) + 2(2-(-1)) - \frac{1}{3}(8-(-1))$

$= \frac{3}{2} + 6 - 3 = 3 + \frac{3}{2} = \frac{9}{2}$

25. The function f is defined as follows.

$$f(x) = \int_1^{x^3} \frac{1}{t}\, dt$$

Find $f'(x)$.

METHOD 1: Use fundamental theorem of calculus (and chain rule)

$$f = \int_1^{u} \frac{1}{t}\, dt \quad \text{and} \quad u = x^3$$

$$\frac{df}{du} = \frac{1}{u} \text{ by Fund Thm} \qquad \frac{du}{dx} = 3x^2$$

$$f'(x) = \frac{df}{dx} = \underbrace{\frac{df}{du}\cdot\frac{du}{dx}}_{\text{CHAIN RULE}} = \frac{1}{u}\cdot 3x^2 = \frac{1}{x^3}\cdot 3x^2 = \boxed{\frac{3}{x}}$$

METHOD 2: Can evaluate integral first

$$f(x) = \int_1^{x^3} \frac{1}{t}\, dt = \Big[\ln|t|\Big]_1^{x^3} = \ln|x^3| - \ln|1|$$

$$= \ln|x^3| - \ln 1 = \ln|x^3| - 0 = \ln|x^3|$$

Then $f'(x) = \frac{1}{x^3}\cdot(x^3)'$ using chain rule

$$= \frac{1}{x^3}\cdot 3x^2 = \boxed{\frac{3}{x}}$$

26. The function f is defined as follows.

$$f(x) = \int_1^{\ln x} e^t \, dt$$

Find $f'(x)$.

METHOD 1. Use fundamental theorem of calculus (and chain rule)

$$f = \int_1^{u} e^t \, dt \quad \text{and} \quad u = \ln x$$

$$\frac{df}{du} = e^u \text{ by Fund Thm} \qquad \frac{du}{dx} = \frac{1}{x}$$

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = e^u \cdot \frac{1}{x} = e^{\ln x} \cdot \frac{1}{x}$$

$$= x \cdot \frac{1}{x} = \boxed{1}$$

METHOD 2. Can evaluate integral first

$$f(x) = \int_1^{\ln x} e^t \, dt = \Big[e^t \Big]_1^{\ln x} = e^{\ln x} - e^1$$

$$= x - e. \quad \text{Therefore } f'(x) = \frac{d}{dx}(x - e)$$

$$= 1 - 0 = \boxed{1}$$