WRITE YOUR NAME:

MAC 2312 Test 2 Thursday March 14th
Total possible score: 18 points

Question 1. Evaluate the integral.

$$\int 7x^6(x^7+11)^3\,dx$$

Sub $u = x^7 + 11 \Rightarrow du = 7x^6\,dx$ $\quad \frac{du}{dx} = 7x^6$

$$\text{Integral} = \int (x^7+11)^3 \cdot 7x^6\,dx$$

$$= \int u^3 \cdot du = \frac{u^4}{4} + C$$

$$= \frac{(x^7+11)^4}{4} + C$$

Question 2. Evaluate the integral.

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\, dx$$

Sub $u = 1 + \sin x \Rightarrow du = \cos x\, dx$

If $x = 0$ then $u = 1 + \sin 0 = 1 + 0 = 1$

If $x = \pi/2$ then $u = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$

$$\int_{x=0}^{x=\pi/2} \frac{1}{1+\sin x} \cdot \cos x\, dx$$

$$= \int_{u=1}^{u=2} \frac{1}{u} \cdot du = \Big[\ln|u|\Big]_{u=1}^{u=2}$$

$$= \ln|2| - \ln|1|$$

$$= \ln 2 - \underbrace{\ln 1}_{0}$$

$$= \ln 2$$

Question 3. Evaluate the integral.

$$\int_0^1 xe^x\,dx$$

Try integration by parts: $\int u\,dv = uv - \int v\,du$

or more specifically $\int_{x=a}^{x=b} u\,dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$

Try $u = x$, $dv = e^x\,dx$ $\left(\frac{dv}{dx} = e^x\right)$ $\Longrightarrow$ $du = 1dx = dx$ $\left(\frac{du}{dx} = 1\right)$, $v = e^x$

$$\int_0^1 \underbrace{x}_{u} \cdot \underbrace{e^x dx}_{dv} = \Big[\underbrace{x}_{u} \cdot \underbrace{e^x}_{v}\Big]_0^1 - \int_0^1 \underbrace{e^x}_{v}\,\underbrace{dx}_{du}$$

$$= \underbrace{1 \cdot e^1}_{1 \cdot e = e} - \underbrace{0 \cdot e^0}_{0} - \int_0^1 e^x\,dx$$

$$= e - \int_0^1 e^x\,dx = e - \Big[e^x\Big]_0^1$$

$$= e - (e^1 - e^0) = e - (e - 1) = e - e + 1$$

$$= 1$$

Question 4. Evaluate the integral.

$\int x^{42} \ln x \, dx$ By parts: $\int u\,dv = uv - \int v\,du$

Try $u = \ln x$, $dv = x^{42}\,dx$ $\Bigg\} \Rightarrow$ $du = \frac{1}{x}\,dx$, $v = \frac{x^{43}}{43}$

$$\text{Integral} = \underbrace{\ln x}_{u} \cdot \underbrace{\frac{x^{43}}{43}}_{v} - \int \underbrace{\frac{x^{43}}{43}}_{v} \cdot \underbrace{\frac{1}{x}\,dx}_{du}$$

$$\frac{x^{43}}{43} \cdot \frac{1}{x} = \frac{1}{43} \cdot \frac{x^{43}}{x} = \frac{1}{43} x^{42}$$

$$= \frac{x^{43} \ln x}{43} - \frac{1}{43} \int x^{42}\,dx$$

$$= \boxed{\frac{x^{43} \ln x}{43} - \frac{1}{43} \cdot \frac{x^{43}}{43} + C} \leftarrow \text{Can stop there}$$

Other ways to write this include $\frac{43x^{43} \ln x - x^{43}}{43^2}$

or $\frac{x^{43}(43 \ln x - 1)}{43^2}$

You can just write 43^2, you don't need to compute it.

$43^2 = (40+3)(40+3) = 1600 + 120 + 120 + 9$
$= 1849$

Question 5. Evaluate the integral.

$$\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx$$

Strategy: Look for "candidates" for du, and remember that even powers of trig-functions are easy to rewrite.

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int_0^{\pi/2} \sin^4 x \, (1 - \sin^2 x) \cos x \, dx$$

A bunch of sines and a single cos works well with $u = \sin x$.

Sub $u = \sin x \Rightarrow du = \cos x \, dx$.

If $x = 0$ then $u = \sin 0 = 0$.

If $x = \pi/2$ then $u = \sin\frac{\pi}{2} = 1$.

$$\text{Integral} = \int_{u=0}^{u=1} u^4 \cdot (1 - u^2) \cdot du$$

$$= \int_0^1 (u^4 - u^6) \, du = \left[\frac{u^5}{5} - \frac{u^7}{7} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{7} = \frac{7}{35} - \frac{5}{35} = \frac{2}{35}$$

Question 6. Evaluate the integral.

$$\int_0^{\sqrt{3}/2} \frac{1}{(1-x^2)^{3/2}}\,dx$$ Looks like trig sub

Try $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

If $x=0$ then $\sin\theta = 0 \Rightarrow \theta = 0$

If $x = \sqrt{3}/2$ then $\sin\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/3$

$$\int_{x=0}^{x=\sqrt{3}/2} \frac{1}{(1-x^2)^{3/2}}\,dx = \int_{\theta=0}^{\theta=\pi/3} \frac{1}{(1-\sin^2\theta)^{3/2}} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{(\cos^2\theta)^{3/2}} \cdot \cos\theta\, d\theta = \int_0^{\pi/3} \frac{1}{\cos^3\theta} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{\cos^2\theta}\, d\theta = \int_0^{\pi/3} \sec^2\theta\, d\theta$$

$$= \left[\tan\theta\right]_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0$$

$$= \sqrt{3} - 0 \quad = \sqrt{3}$$

Question 7. Evaluate the integral.

$$\int_5^6 \frac{3x-10}{x^2-7x+12}\,dx$$ Looks like partial fractions

Denominator $x^2-7x+12$ factors as $(x-3)(x-4)$

Want $\dfrac{3x-10}{(x-3)(x-4)} = \dfrac{A}{x-3} + \dfrac{B}{x-4}$

$\downarrow$ multiply both sides by $(x-3)(x-4)$

$$\begin{aligned} 3x-10 &= A(x-4) + B(x-3) \\ &= Ax - 4A + Bx - 3B \\ &= (A+B)x + (-4A-3B) \end{aligned}$$

$$\Rightarrow \begin{aligned} A+B &= 3 \\ -4A-3B &= -10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4A+4B &= 12 \\ -4A-3B &= -10 \\ \hline B &= 2 \end{aligned} \quad \Rightarrow A = 1$$

$$\int_5^6 \frac{3x-10}{(x-3)(x-4)}\,dx = \int_5^6 \left(\frac{1}{x-3} + \frac{2}{x-4}\right)dx$$

$$= \Big[\ln|x-3| + 2\ln|x-4|\Big]_5^6$$

$$= \ln(6-3) + 2\ln(6-4) \underset{\substack{\uparrow \\ \text{NOTE}}}{-} \left(\ln(5-3) \underset{\substack{\uparrow \\ \text{NOTE}}}{+} 2\ln(5-4)\right)$$

$$= \ln 3 + 2\ln 2 - \ln 2 - 2\underbrace{\ln 1}_{0}$$

$$= \ln 3 + \ln 2 \quad \text{or} \quad \ln(3\cdot 2) = \ln 6$$

Question 8. Determine whether the integral converges or diverges, and find its value if it converges.

$$\int_9^\infty \frac{1}{\sqrt{x}}\,dx$$

Consider $\int_9^M \frac{1}{\sqrt{x}}\,dx = \int_9^M x^{-1/2}\,dx$

$= \left[\frac{x^{1/2}}{1/2}\right]_9^M = 2\left[x^{1/2}\right]_9^M = 2(M^{1/2} - 9^{1/2})$

$= 2(\sqrt{M} - 3)$ or $2\sqrt{M} - 6$

When $M \to \infty$, $2\sqrt{M} - 6$ approaches infinity.

Integral DIVERGES.

Question 9. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{83}{n}$$

$$\frac{83}{1} + \frac{83}{2} + \frac{83}{3} + \frac{83}{4} + \frac{83}{5} + \cdots$$

$$= 83\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right)$$

or more briefly, $\displaystyle\sum_{n=1}^{\infty} \frac{83}{n} = \sum_{n=1}^{\infty} 83 \cdot \frac{1}{n} = 83 \sum_{n=1}^{\infty} \frac{1}{n}$

This is a constant multiple of the famous harmonic series, which we know diverges, so the given series DIVERGES.