WRITE YOUR NAME:

MAC 2312 Section U03 Test 2
Friday March 3rd
Total possible score: 20 points (2 points per page)

Question 1. Evaluate the integral.

$$\int_2^3 \sqrt{9x-2}\,dx$$

Substitute $u = 9x-2$ $\quad \frac{du}{dx} = 9$

$\Rightarrow du = 9\,dx$

$\frac{1}{9}du = dx$

If $x=2$, then $u = 9\cdot 2 - 2 = 18-2 = 16$

If $x=3$, then $u = 9\cdot 3 - 2 = 27-2 = 25$

$$\int_{x=2}^{x=3} \sqrt{9x-2}\,dx = \int_{u=16}^{u=25} \sqrt{u}\cdot\frac{1}{9}\,du = \frac{1}{9}\int_{16}^{25} u^{1/2}\,du$$

$$= \frac{1}{9}\left[\frac{u^{3/2}}{3/2}\right]_{16}^{25} = \frac{1}{9}\cdot\frac{2}{3}\left[u^{3/2}\right]_{16}^{25}$$

$$= \frac{2}{27}\left(25^{3/2} - 16^{3/2}\right) = \frac{2}{27}\left((25^{1/2})^3 - (16^{1/2})^3\right)$$

$$= \underbrace{\frac{2}{27}\left(5^3 - 4^3\right)}_{\text{stopping here is OK}} = \frac{2}{27}(125-64) = \frac{2}{27}\cdot 61$$

$$= 122/27$$

Question 2. Evaluate the integral.

$$\int_0^{\pi/6} \sin^4 x \cos x \, dx$$

Substitute $u = \sin x$

$\Rightarrow du = \cos x \, dx$

If $x = 0$, then $u = \sin 0 = 0$

If $x = \frac{\pi}{6}$, then $u = \sin \frac{\pi}{6} = \frac{1}{2}$

$$\int_{x=0}^{x=\pi/6} \sin^4 x \cos x \, dx = \int_{u=0}^{u=1/2} u^4 \, du$$

$$= \left[\frac{u^5}{5} \right]_0^{1/2} = \frac{1}{5} \left[u^5 \right]_0^{1/2} = \frac{1}{5} \cdot \frac{1}{2^5}$$

$$= \frac{1}{5} \cdot \frac{1}{32} = \frac{1}{160}$$

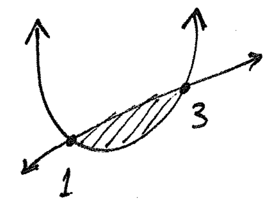
Question 3. Find the **area** of the region bounded by $y = x^2$ and $y = 4x - 3$.

Intersection points? $x^2 = 4x - 3$

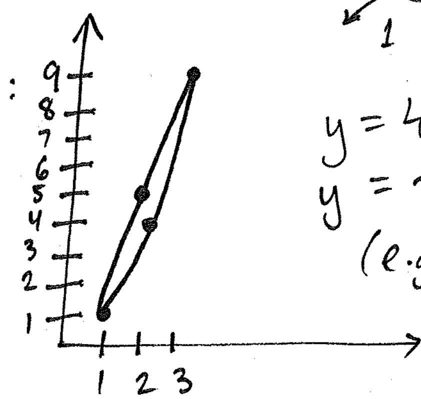
$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

So $1 \leq x \leq 3$. Rough picture:



More careful picture:



$y = 4x - 3$ is top
$y = x^2$ is bottom
(e.g. plug in $x = 2$)

$$\text{Area} = \int_1^3 (\text{top} - \text{bottom})\, dx = \int_1^3 (4x - 3 - x^2)\, dx$$

$$= \left[\frac{4x^2}{2} - 3x - \frac{x^3}{3} \right]_1^3 = \left[2x^2 - 3x - \frac{x^3}{3} \right]_1^3$$

$$= 2\left[x^2\right]_1^3 - 3\left[x\right]_1^3 - \frac{1}{3}\left[x^3\right]_1^3$$

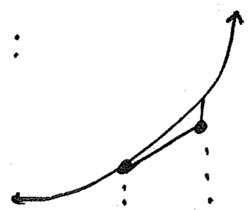
$$= 2(9-1) - 3(3-1) - \frac{1}{3}(27-1)$$

$$= 2 \cdot 8 - 3 \cdot 2 - \frac{26}{3} = 16 - 6 - \frac{26}{3} = \frac{30}{3} - \frac{26}{3} = \frac{4}{3}$$

Question 4. Find the **area** of the region bounded by $y = e^x$, $y = 1 + x$, and $x = 1$.

Notice both $y = e^x$ and $y = 1 + x$ go through the point $(0, 1)$.

Rough picture:



$$\text{Area} = \int_0^1 (\text{top} - \text{bottom})\, dx = \int_0^1 (e^x - (1+x))\, dx$$

$$= \int_0^1 (e^x - 1 - x)\, dx = \left[e^x - x - \frac{x^2}{2} \right]_0^1$$

$$= \left(e^1 - 1 - \frac{1}{2} \right) - \left(e^0 - 0 - \frac{0}{2} \right)$$

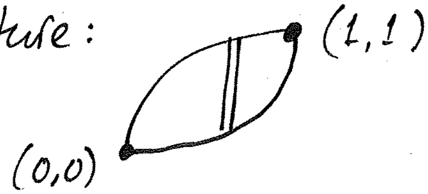
$$= e - 1 - \frac{1}{2} - 1 = e - \frac{5}{2}$$

or $\dfrac{2e-5}{2}$

Question 5. Let A be the region bounded by $y = x^3$ and $y = \sqrt{x}$. Find the volume obtained when the region A is revolved around the line $y = -1$.

The two curves $y=x^3$ and $y=\sqrt{x}$ intersect at $x=0$ and $x=1$.

Rough picture:

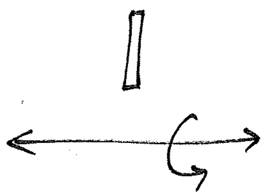


$y=\sqrt{x}$ is top

$y=x^3$ is bottom

(remember $0 \le x \le 1$)

Revolve around line $y=-1$ (horizontal line)



WASHERS R = top curve − axis of revolution

$= \sqrt{x} - (-1) = \sqrt{x} + 1$

r = bottom curve − axis of revolution

$= x^3 - (-1) = x^3 + 1$

$$\text{Volume} = \int_0^1 (\pi R^2 - \pi r^2)\,dx = \pi \int_0^1 (R^2 - r^2)\,dx$$

$$= \pi \int_0^1 \left((\sqrt{x}+1)^2 - (x^3+1)^2 \right) dx = \pi \int_0^1 \left(x + 2\sqrt{x} + 1 - (x^6 + 2x^3 + 1)\right) dx$$

$$= \pi \int_0^1 \left(x + 2\sqrt{x} - x^6 - 2x^3 \right) dx = \pi \int_0^1 \left(x + 2x^{1/2} - x^6 - 2x^3\right) dx$$

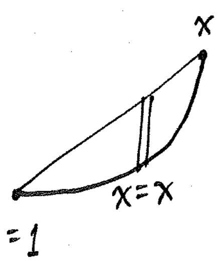
$$= \pi \left[\frac{x^2}{2} + \frac{2x^{3/2}}{3/2} - \frac{x^7}{7} - \frac{2x^4}{4} \right]_0^1 = \pi \left(\frac{1}{2} + \frac{4}{3} - \frac{1}{7} - \frac{1}{2} \right)$$

$$= \pi \left(\frac{4}{3} - \frac{1}{7} \right) = \pi \left(\frac{28}{21} - \frac{3}{21} \right) = \frac{25\pi}{21}$$

Question 6. Let A be the region bounded by $y = x^2$ and $y = 4x - 3$. Find the volume obtained when the region A is revolved around the y-axis.

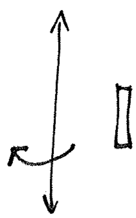
This region appeared in Question 3!

Rough picture:



$y = 4x - 3$ is top

$y = x^2$ is bottom



SHELLS $\quad r = \text{slice} - \text{axis of revolution}$

$= x - 0 = x$

$h = \text{top curve} - \text{bottom curve}$

$= 4x - 3 - x^2$

$$\text{Volume} = \int_1^3 2\pi r h \, dx = 2\pi \int_1^3 x(4x - 3 - x^2)\, dx$$

$$= 2\pi \int_1^3 (4x^2 - 3x - x^3)\, dx = 2\pi \left[4\frac{x^3}{3} - \frac{3x^2}{2} - \frac{x^4}{4} \right]_1^3$$

$$= 2\pi \left(\frac{4}{3}[x^3]_1^3 - \frac{3}{2}[x^2]_1^3 - \frac{1}{4}[x^4]_1^3 \right)$$

$$= 2\pi \left(\frac{4}{3}(27-1) - \frac{3}{2}(9-1) - \frac{1}{4}(81-1) \right)$$

$$= 2\pi \left(\frac{4}{3}\cdot 26 - \frac{3}{2}\cdot 8 - \frac{1}{4}\cdot 80 \right) = 2\pi \left(\frac{104}{3} - 12 - 20 \right)$$

$$= 2\pi \left(\frac{104}{3} - 32 \right) = 2\pi \left(\frac{104}{3} - \frac{96}{3} \right) = 2\pi \cdot \frac{8}{3} = \frac{16\pi}{3}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \sqrt{1 + (y')^2}\, dx$$

Question 7. Find the length of the curve $y = 2x^{3/2}$ between $x = 0$ and $x = 11$.

$$\text{Length of curve} = \int_{x=0}^{x=11} ds$$

$$= \int_{x=0}^{x=11} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If $y = 2x^{3/2}$ then $\frac{dy}{dx} = 2 \cdot \frac{3}{2} x^{1/2} = 3x^{1/2}$

$$\text{Length of curve} = \int_0^{11} \sqrt{1 + (3x^{1/2})^2}\, dx$$

$$= \int_0^{11} \sqrt{1 + 9x}\, dx$$

Now try substitution

Sub $u = 1 + 9x$

$\Rightarrow du = 9\, dx$

$\frac{1}{9} du = dx$

If $x = 0$, then $u = 1 + 9 \cdot 0 = 1$

If $x = 11$, then $u = 1 + 9 \cdot 11 = 100$

Length

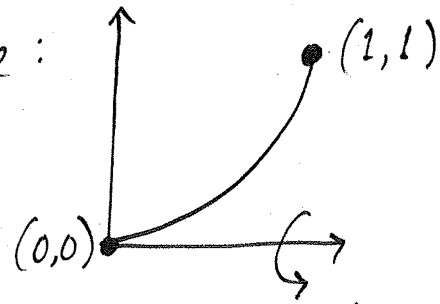
$$= \int_{u=1}^{u=100} \sqrt{u} \cdot \frac{1}{9}\, du = \frac{1}{9} \int_1^{100} u^{1/2}\, du$$

$$= \frac{1}{9} \left[\frac{u^{3/2}}{3/2} \right]_1^{100} = \frac{1}{9} \cdot \frac{2}{3} \left[u^{3/2} \right]_1^{100} = \frac{2}{27} \left(100^{3/2} - 1 \right)$$

$$= \frac{2}{27} (10^3 - 1) = \frac{2}{27} \cdot 999 = \frac{2}{3} \cdot 111 = \frac{222}{3} = 74$$

Question 8. Let C be the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$. Find the surface area obtained by revolving C around the x-axis.

Rough picture:



$$\text{Surface area} = \int_0^1 2\pi r\, ds = \int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^1 2\pi \cdot x^3 \cdot \sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi \int_0^1 x^3\sqrt{1+9x^4}\,dx$$

Try $u = 1+9x^4$

$du = 36x^3\,dx$

$\frac{1}{36}\,du = x^3\,dx$

If $x=0$ then $u = 1+0=1$

If $x=1$ then $u=1+9=10$

Surface area

$$= 2\pi \int_{u=1}^{u=10} \sqrt{u}\cdot\frac{1}{36}\,du$$

$$= \frac{\pi}{18}\int_1^{10} u^{1/2}\,du = \frac{\pi}{18}\left[\frac{u^{3/2}}{3/2}\right]_1^{10}$$

$$= \frac{\pi}{18}\cdot\frac{2}{3}\left(10^{3/2}-1\right) = \frac{\pi}{27}\left(10^{3/2}-1\right)$$

$$\int u\,dv = uv - \int v\,du$$

Question 9: Evaluate the integral.

$$\int_{x=a}^{x=b} u\,dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

$$\int_1^e x^7 \ln x \, dx$$

Try integration by parts

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

$$dv = x^7\,dx \quad \Rightarrow \quad v = \frac{x^8}{8}$$

$$\int_1^e \underbrace{\ln x}_{u} \cdot \underbrace{x^7 dx}_{dv} = \Big[\underbrace{\ln x}_{u} \cdot \underbrace{\frac{x^8}{8}}_{v}\Big]_1^e - \int_1^e \underbrace{\frac{x^8}{8}}_{v} \cdot \underbrace{\frac{1}{x}\,dx}_{du}$$

$$= \frac{1}{8}\Big[x^8 \ln x\Big]_1^e - \frac{1}{8}\int_1^e x^7\,dx$$

$$= \frac{1}{8}\Big(e^8 \underbrace{\ln e}_{1} - 1\underbrace{\ln 1}_{0}\Big) - \frac{1}{8}\Big[\frac{x^8}{8}\Big]_1^e$$

$$= \frac{e^8}{8} - \frac{1}{64}\left(e^8 - 1\right) = \frac{8e^8}{64} - \frac{e^8 - 1}{64}$$

$$= \frac{7e^8 + 1}{64}$$

Question 10. A force of $F(x) = 500 - 3x^2$ newtons (applied in the positive x direction) moves an object 8 meters, from $x = 2$ to $x = 10$. Find the total work done by the force on the object.

$$\text{Work} = \int_a^b F(x)\,dx = \int_2^{10} (500 - 3x^2)\,dx$$

$$= \Big[500x - x^3 \Big]_2^{10}$$

$$= 500\big[x\big]_2^{10} - \big[x^3\big]_2^{10}$$

$$= 500(10-2) - (1000-8)$$

$$= 500 \cdot 8 - 992$$

$$= 4000 - 992$$

$$= 3008$$