MAC2312 Section U03
Suggested problems for Test 2
(Test 2 is Friday March 3rd, in class)

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1. Find the area bounded by the curves $y = x^2$ and $y = 2x$.

Intersection points? $x^2 = 2x$

$$x^2 - 2x = 0$$

$$x(x-2) = 0 \Rightarrow x = 0, \; x = 2$$

$$\text{Area} = \int_0^2 (\text{top} - \text{bottom})\, dx = \int_0^2 (2x - x^2)\, dx$$

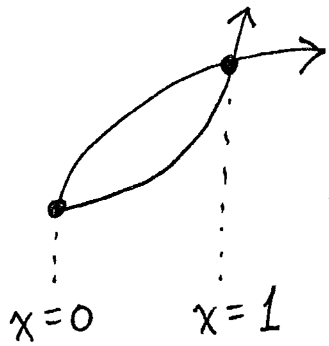
$$= \left[x^2 - \frac{x^3}{3} \right]_0^2 = 2^2 - \frac{2^3}{3} = 4 - \frac{8}{3}$$

$$= \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

2. Let R be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Find the volume obtained by revolving R around the x-axis.

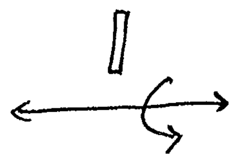
Intersection points? $x^2 = \sqrt{x} \Rightarrow (x^2)^2 = (\sqrt{x})^2$

$$x^4 = x$$
$$x^4 - x = 0$$
$$x(x^3-1) = 0$$
$$x = 0 \text{ or } x^3 - 1 = 0$$
$$x^3 = 1$$
$$x = 1$$

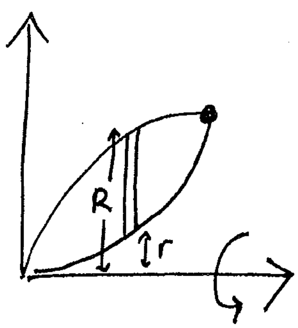


$y = \sqrt{x}$ TOP

$y = x^2$ BOTTOM



WASHERS



$$\text{Volume} = \int (\pi R^2 - \pi r^2)\,dx$$

$R = (\text{top curve}) - (\text{axis}) = \sqrt{x} - 0 = \sqrt{x}$

$\quad\; y = \sqrt{x} \qquad y = 0$

$r = (\text{bottom curve}) - (\text{axis}) = x^2 - 0 = x^2$

$\quad\; y = x^2 \qquad y = 0$

$$\text{Volume} = \pi \int_0^1 (R^2 - r^2)\,dx = \pi \int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right) dx$$

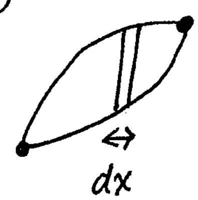
$$= \pi \int_0^1 (x - x^4)\,dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right)$$

$$= \pi \left(\frac{5}{10} - \frac{2}{10} \right) = \frac{3\pi}{10}$$

3. Let R be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Find the volume obtained by revolving R around the y-axis.

Intersection points: $x = 0$, $x = 1$

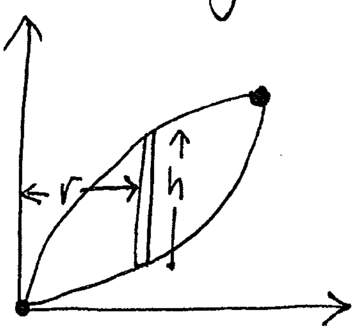
If we keep these curves as $y = f(x)$ then we're slicing like this:



Revolve around y-axis:

SHELLS

$$\text{Volume} = \int 2\pi r h \, dx \qquad r = ? \quad h = ?$$



$$h = (\text{top curve}) - (\text{bottom curve})$$
$$\qquad\quad y = \sqrt{x} \qquad\qquad y = x^2$$

$$h = \sqrt{x} - x^2 = x^{1/2} - x^2$$

$$r = \underset{x = x}{(\text{typical slice})} - \underset{x=0}{(\text{axis})} = x - 0 = x$$

$$\text{Volume} = 2\pi \int_0^1 r h \, dx = 2\pi \int_0^1 x\left(x^{1/2} - x^2\right) dx$$

$$= 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx = 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^4}{4} \right]_0^1$$

$$= 2\pi \left(\frac{2}{5} - \frac{1}{4} \right) = 2\pi \left(\frac{8}{20} - \frac{5}{20} \right) = \frac{3\pi}{10}$$

4. Find the length of the curve $y = x^{3/2}$ between $x = 0$ and $x = 1$.

$$\text{Length} = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$\text{Length} = \int_0^1 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx = \int_0^1 \sqrt{1 + \tfrac{9}{4}x}\, dx$$

Sub $u = 1 + \frac{9}{4}x$

$du = \frac{9}{4} dx$

$\frac{4}{9} du = dx$

If $x=0$, then $u=1$

If $x=1$, then $u = 1 + \frac{9}{4} = \frac{13}{4}$

$$\text{Integral} = \int_1^{13/4} \sqrt{u} \cdot \frac{4}{9}\, du$$

$$= \frac{4}{9} \int_1^{13/4} u^{1/2}\, du = \frac{4}{9} \left[\frac{u^{3/2}}{3/2} \right]_1^{13/4}$$

$$= \frac{4}{9} \cdot \frac{2}{3} \left(\underbrace{\left(\frac{13}{4}\right)^{3/2}}_{\frac{13^{3/2}}{4^{3/2}}} - \underbrace{1^{3/2}}_{=1} \right)$$

Can stop there, or can simplify a little.

$$\frac{8}{27}\left(\frac{13^{3/2}}{8} - 1 \right)$$

5. Find the length of the curve $y = x^{2/3}$ between $x = 0$ and $x = 1$.

$$\text{Length} = \int ds = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = x^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-1/3}$$

$$\text{Length} = \int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}\, dx = \int_0^1 \sqrt{1+\frac{4}{9x^{2/3}}}\, dx$$

$$= \int_0^1 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx = \int_0^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\, dx$$

$$= \frac{1}{3}\int_0^1 \sqrt{9x^{2/3}+4}\cdot x^{-1/3}\, dx$$

Sub $u = 9x^{2/3}+4$

$du = 9\cdot\frac{2}{3}x^{-1/3}\,dx$

$du = 6x^{-1/3}\,dx$

$\frac{1}{6}du = x^{-1/3}\,dx$

When $x=0$, $u = 9\cdot 0+4 = 4$

When $x=1$, $u = 9\cdot 1+4 = 13$

$$\text{Integral} = \frac{1}{3}\int_4^{13}\sqrt{u}\cdot\frac{1}{6}\,du = \frac{1}{18}\int_4^{13} u^{1/2}\,du$$

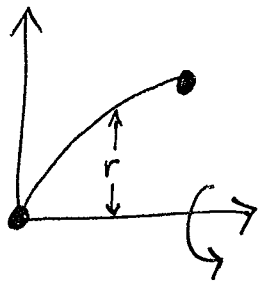
$$= \frac{1}{18}\left[\frac{u^{3/2}}{3/2}\right]_4^{13} = \frac{1}{18}\cdot\frac{2}{3}\left(13^{3/2}-4^{3/2}\right)$$

Can stop there, or can simplify

$$\frac{1}{27}\left(13^{3/2}-8\right)$$

6. Let C be the portion of the curve $y = x^{1/2}$ between $x = 0$ and $x = 1$. Find the surface area obtained by revolving C around the x-axis.

Surface area $= \displaystyle\int 2\pi r \, ds$



$$r = f(x) = x^{1/2} = \sqrt{x}$$

$$ds = \sqrt{1+(f'(x))^2}\, dx = \sqrt{1+\left(\tfrac{1}{2}x^{-1/2}\right)^2}\, dx = \sqrt{1+\tfrac{1}{4x}}\, dx$$

Surface area $= 2\pi \displaystyle\int_0^1 r\, ds = 2\pi \int_0^1 \sqrt{x}\cdot\sqrt{1+\tfrac{1}{4x}}\, dx$

$$= 2\pi \int_0^1 \sqrt{x\cdot\left(1+\tfrac{1}{4x}\right)}\, dx = 2\pi \int_0^1 \sqrt{x+\tfrac{1}{4}}\, dx$$

Now can either do $u = x + \frac{1}{4}$ or just "know" antiderivative of $\left(x+\frac{1}{4}\right)^{1/2}$

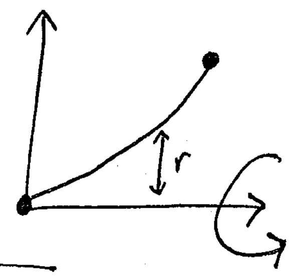
$$2\pi\left[\frac{\left(x+\frac{1}{4}\right)^{3/2}}{3/2}\right]_0^1 = 2\pi\cdot\frac{2}{3}\left(\left(1+\tfrac{1}{4}\right)^{3/2} - \left(0+\tfrac{1}{4}\right)^{3/2}\right)$$

$$= \frac{4\pi}{3}\left(\left(\tfrac{5}{4}\right)^{3/2} - \left(\tfrac{1}{4}\right)^{3/2}\right)$$

Can stop here, or can simplify. Denominator $4^{3/2}$ is 8

7. Let C be the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$. Find the surface area obtained by revolving C around the x-axis.

Surface area $= \int 2\pi r \, ds$



$r = f(x) = x^3$

$f'(x) = 3x^2$ so $ds = \sqrt{1+(f'(x))^2}\, dx$

$= \sqrt{1+(3x^2)^2}\, dx = \sqrt{1+9x^4}\, dx$

Surface area $= 2\pi \int_0^1 r \, ds = 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\, dx$

Then sub $u = 1+9x^4$ If $x=0$, then $u = 1$

$du = 36x^3 dx$ If $x=1$, then $u = 10$

$\frac{1}{36} du = x^3 dx$

Integral $= 2\pi \int_1^{10} \sqrt{u} \cdot \frac{1}{36} du = \frac{2\pi}{36} \int_1^{10} u^{1/2} du$

$= \frac{\pi}{18} \left[\frac{u^{3/2}}{3/2} \right]_1^{10} = \frac{\pi}{18} \cdot \frac{2}{3} \cdot \left(10^{3/2} - 1 \right)$

$= \frac{\pi}{27} \left(10^{3/2} - 1 \right)$

8. Evaluate the integral.

$$\int x^2 e^x \, dx$$

Integration by parts: $\int u\,dv = uv - \int v\,du$

Try $u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^x dx \Rightarrow v = e^x$

$$I = \int x^2 e^x dx = x^2 e^x - \underbrace{\int e^x 2x\,dx}_{\text{Integrate by parts again}}$$

This time $u = 2x \Rightarrow du = 2dx$

$dv = e^x dx \Rightarrow v = e^x$

$$I = x^2 e^x - \left(2xe^x - \int e^x 2dx \right)$$

$$I = x^2 e^x - 2xe^x + \int 2e^x dx$$

$$I = \underline{x^2 e^x - 2xe^x + 2e^x} \quad (+c)$$

9. Evaluate the integral.

$$\int_0^{\pi/2} x \cos x \, dx$$

For integration by parts with <u>definite</u> integral, can do

$$\int_{x=a}^{x=b} u \, dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v \, du$$

Try $u = x \Rightarrow du = 1\,dx$

$dv = \cos x \, dx \Rightarrow v = \sin x$

$$I = \int_0^{\pi/2} x \cos x \, dx = \Big[x \sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x \cdot 1 \, dx$$

$$= \underbrace{\Big(\tfrac{\pi}{2} \underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{0 \sin 0}_{0}\Big)}_{\frac{\pi}{2}} - \int_0^{\pi/2} \sin x \, dx$$

$$= \frac{\pi}{2} - \int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \Big[-\cos x\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \Big[\cos x\Big]_0^{\pi/2} = \frac{\pi}{2} + \Big(\underbrace{\cos \tfrac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1}\Big)$$

$$= \frac{\pi}{2} - 1$$

$$\int u\,dv = uv - \int v\,du$$

10. Evaluate the integral.

$$\int x^7 \ln x \, dx$$

Try $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = x^7 dx \Rightarrow v = \frac{x^8}{8}$

$$I = \int x^7 \ln x \, dx = \ln x \cdot \frac{x^8}{8} - \int \frac{x^8}{8} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^8 \ln x}{8} - \frac{1}{8}\int x^7 dx$$

$$= \frac{x^8 \ln x}{8} - \frac{1}{8} \cdot \frac{x^8}{8} \qquad (+c)$$

Can stop there, or can write other ways

$$\frac{x^8 \ln x}{8} - \frac{x^8}{64} \qquad \frac{8x^8 \ln x - x^8}{64}$$

$$\frac{x^8(8\ln x - 1)}{64}$$

11. A force of $F(x) = 10 - 2x$ newtons (applied in the positive x direction) moves an object 3 meters, from $x = 2$ to $x = 5$. Find the total work done by the force on the object.

$$\text{Work} = \int_2^5 F(x)\,dx = \int_2^5 (10 - 2x)\,dx$$

$$= \Big[\, 10x - x^2 \,\Big]_2^5 = \Big[10x\Big]_2^5 - \Big[x^2\Big]_2^5$$

$$= 10\Big[x\Big]_2^5 + \Big[x^2\Big]_5^2$$

$$= 10(5-2) + (2^2 - 5^2)$$

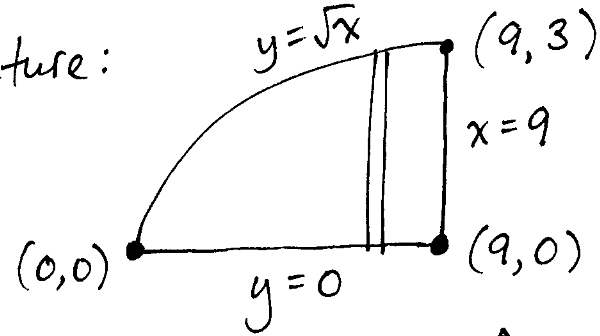
$$= 10 \cdot 3 + 4 - 25$$

$$= 30 + 4 - 25$$

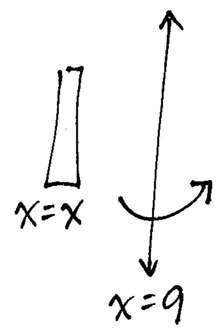
$$= 9 \text{ newton-meters or joules}$$

12. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $x = 9$.

Rough picture:



Revolved around $x=9$:



SHELLS. $r = \text{axis} - \text{slice} = 9 - x$

$h = \text{top} - \text{bottom} = \sqrt{x} - 0 = \sqrt{x}$

$$\text{Volume} = \int_0^9 2\pi r h \, dx = 2\pi \int_0^9 r h \, dx = 2\pi \int_0^9 (9-x)\sqrt{x}\, dx$$

$$= 2\pi \int_0^9 \left(9x^{1/2} - x^{3/2}\right) dx = 2\pi \left[9\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} \right]_0^9$$

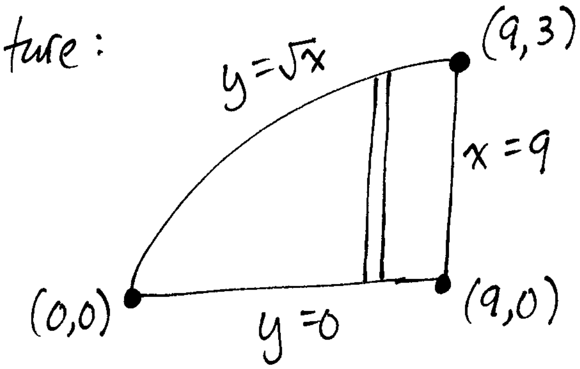
$$= 2\pi \left[6x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^9 = 2\pi\left(6 \cdot 9^{3/2} - \frac{2}{5} \cdot 9^{5/2}\right)$$

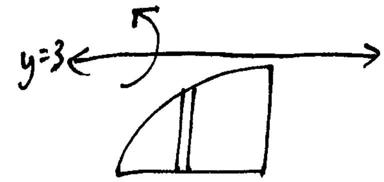
$$= 2\pi\left(6 \cdot 3^3 - \frac{2}{5} \cdot 3^5\right)$$ I would just leave it in that form.

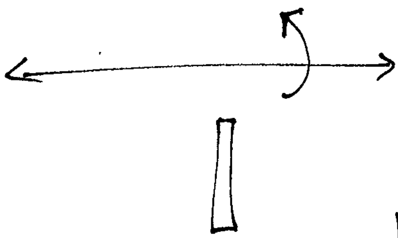
$$2\pi\left(6 \cdot 27 - \frac{2}{5} \cdot 243\right) = \ldots = \frac{648\pi}{5}$$

13. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 9$ is revolved about the line $y = 3$.

Rough picture:







WASHERS

$r = \underset{\{y=3\}}{\text{axis}} - \underset{\{y=\sqrt{x}\}}{\text{top curve}} = 3-\sqrt{x}$

$R = \underset{\{y=3\}}{\text{axis}} - \underset{\{y=0\}}{\text{bottom curve}} = 3-0=3$

$$\text{Volume} = \int_0^9 (\pi R^2 - \pi r^2)\,dx = \pi \int_0^9 (R^2 - r^2)\,dx$$

$$= \pi \int_0^9 \left(3^2 - (3-\sqrt{x})^2\right) dx = \pi \int_0^9 \left(9 - (9 - 6\sqrt{x} + x)\right) dx$$

$$= \pi \int_0^9 (6\sqrt{x} - x)\,dx = \pi \left[\frac{6x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^9$$

$$= \pi \left[4x^{3/2} - \frac{x^2}{2} \right]_0^9 = \pi \left(4 \cdot 9^{3/2} - \frac{9^2}{2} \right)$$

$$= \pi \left(4 \cdot 3^3 - \frac{81}{2} \right) = \pi \left(4 \cdot 27 - \frac{81}{2} \right) = \pi \left(\frac{216}{2} - \frac{81}{2} \right)$$

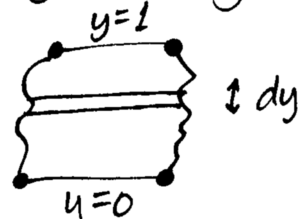
$$= 135\pi/2$$

14. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $x = -1$.

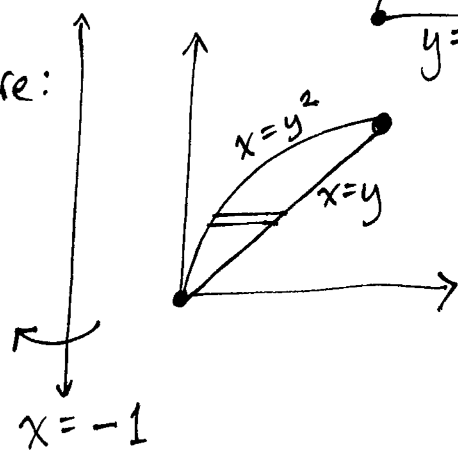
$x = f(y)$ Left curve, right curve

Intersection points? $y^2 = y$ $\quad y^2 - y = 0 \quad y(y-1) = 0$

$y = 0, y = 1$. "Lying" picture:

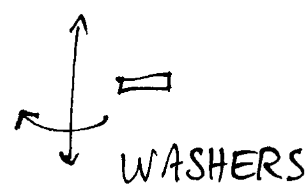


More accurate picture:



$x = y^2$ is left curve

$x = y$ is right curve



r = left curve − axis $= y^2 - (-1) = y^2 + 1$

R = right curve − axis $= y - (-1) = y + 1$

$$\text{Volume} = \int_0^1 (\pi R^2 - \pi r^2)\,dy = \pi \int_0^1 (R^2 - r^2)\,dy$$

$$= \pi \int_0^1 \left((y+1)^2 - (y^2+1)^2 \right) dy = \pi \int_0^1 \left(y^2 + 2y + 1 - (y^4 + 2y^2 + 1) \right) dy$$

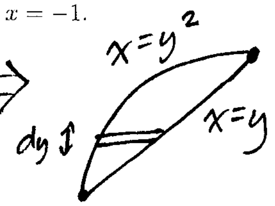
$$= \pi \int_0^1 (2y - y^2 - y^4)\,dy = \pi \left[y^2 - \frac{y^3}{3} - \frac{y^5}{5} \right]_0^1$$

$$= \pi \left(1 - \frac{1}{3} - \frac{1}{5}\right) = \pi \left(\frac{15}{15} - \frac{5}{15} - \frac{3}{15} \right) = 7\pi/15$$

ALTERNATE Solution to #14
where we rewrite as $y = f(x)$

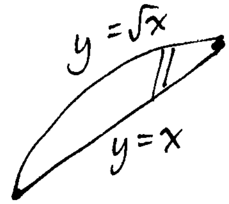
14. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $x = -1$.

$x = y^2$ and $x = y \Rightarrow$

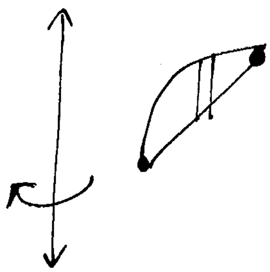


$x = y^2$: left curve
$x = y$: right curve

Can describe same region as:



$y = \sqrt{x}$: top curve
$y = x$: bottom curve



$x = -1$

SHELLS

$$r = \text{slice} - \text{axis} = x - (-1) = x + 1$$

$$h = \text{top} - \text{bottom} = \sqrt{x} - x$$

$$\text{Volume} = \int_0^1 2\pi r h \, dx = 2\pi \int_0^1 r h \, dx$$

$$= 2\pi \int_0^1 (x+1)(\sqrt{x} - x)\, dx = 2\pi \int_0^1 \left(x^{3/2} - x^2 + x^{1/2} - x\right) dx$$

$$= 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^3}{3} + \frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1$$

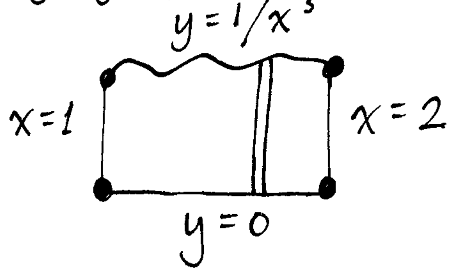
$$= 2\pi \left[\frac{2}{5} x^{5/2} - \frac{x^3}{3} + \frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1$$

$$= 2\pi \left(\frac{2}{5} - \frac{1}{3} + \frac{2}{3} - \frac{1}{2} \right)$$

$$= 2\pi \left(\frac{12}{30} - \frac{10}{30} + \frac{20}{30} - \frac{15}{30} \right) = 2\pi \cdot \frac{7}{30} = \frac{7\pi}{15}$$

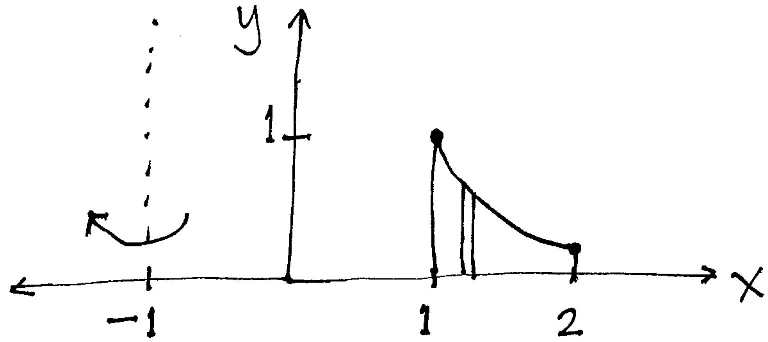
15. Find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

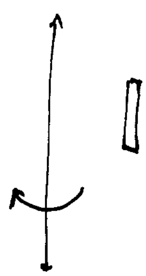
"Lying" picture might be good enough:



Note: Had to know $1/x^3$ is larger than 0 to conclude $\frac{1}{x^3}$ is "top curve"

Slightly better picture:





SHELLS.

$r = \text{slice} - \text{axis} = x - (-1) = x + 1$

$h = \text{top} - \text{bottom} = \frac{1}{x^3} - 0 = x^{-3}$

$$\text{Volume} = \int_1^2 2\pi r h \, dx = 2\pi \int_1^2 (x+1)x^{-3}\, dx = 2\pi \int_1^2 (x^{-2} + x^{-3})\, dx$$

$$= 2\pi \left[\frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} \right]_1^2 = 2\pi \left[-\frac{1}{x} - \frac{1}{2x^2} \right]_1^2$$

$$= 2\pi \left[\frac{1}{x} + \frac{1}{2x^2} \right]_2^1 = 2\pi \left(1 + \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{8} \right) \right) = 2\pi \cdot \frac{7}{8}$$

$$= \frac{7\pi}{4}$$

16. Evaluate the integral.

$$\int_1^2 \sqrt{5x-1}\, dx$$

Try substituting $u = 5x - 1$

$\Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}du = dx$

If $x=1$, then $u = 5\cdot 1 - 1 = 5-1 = 4$

If $x=2$, then $u = 5\cdot 2 - 1 = 10-1 = 9$

$$\int_{x=1}^{x=2} \sqrt{5x-1}\, dx = \int_{u=4}^{u=9} \sqrt{u} \cdot \frac{1}{5}\, du$$

$$= \frac{1}{5}\int_4^9 u^{1/2}\, du = \frac{1}{5}\left[\frac{u^{3/2}}{3/2}\right]_4^9$$

$$= \frac{1}{5}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^9 = \frac{2}{15}\left(9^{3/2} - 4^{3/2}\right)$$

$$= \frac{2}{15}\left(\left(9^{1/2}\right)^3 - \left(4^{1/2}\right)^3\right) = \frac{2}{15}\left(3^3 - 2^3\right)$$

$$= \frac{2}{15}(27-8) = \frac{2}{15}\cdot 19 = \frac{38}{15}$$

17. Evaluate the integral.

$$\int_{\pi/2}^{\pi} 6 \sin x(\cos x + 1)^5 \, dx$$

Try substituting $u = \cos x + 1$

$$\Rightarrow du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

If $x = \frac{\pi}{2}$, then $u = \cos\frac{\pi}{2} + 1 = 0 + 1 = 1$

If $x = \pi$, then $u = \cos\pi + 1 = -1 + 1 = 0$

$$\text{Integral} = \int_{x=\pi/2}^{x=\pi} 6 \cdot (\cos x + 1)^5 \cdot \sin x \, dx$$

$$= \int_{u=1}^{u=0} 6 \cdot u^5 \cdot (-du) = -6\int_1^0 u^5 \, du$$

$$= 6\int_0^1 u^5 \, du = 6 \cdot \left[\frac{u^6}{6}\right]_0^1$$

$$= 6 \cdot \frac{1}{6} = 1$$

18. Evaluate the integral.

$$\int_1^3 \frac{x+2}{\sqrt{x^2+4x+4}}\,dx$$

Try substituting $u = x^2 + 4x + 4$

$$\Rightarrow \frac{du}{dx} = 2x + 4 \quad \Rightarrow du = (2x+4)\,dx$$

$$\Rightarrow \frac{1}{2}\,du = (x+2)\,dx$$

If $x = 1$, then $u = 1^2 + 4\cdot 1 + 4 = 1 + 4 + 4 = 9$

If $x = 3$, then $u = 3^2 + 4\cdot 3 + 4 = 9 + 12 + 4 = 25$

$$\text{Integral} = \int_{x=1}^{x=3} \frac{1}{\sqrt{x^2+4x+4}}\,(x+2)\,dx$$

$$= \int_{u=9}^{u=25} \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_9^{25} u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]_9^{25} = \left[u^{1/2}\right]_9^{25} = 25^{1/2} - 9^{1/2}$$

$$= 5 - 3 = 2$$