WRITE YOUR NAME:

MAC 2312 Test 3 Thursday April 11th
Total possible score: 18 points

Question 1. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n+17}$$

Limit of n^{th} term is

$$\lim_{n\to\infty} \frac{n}{n+17} = \lim_{n\to\infty} \frac{n}{n+17} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{17}{n}}$$

$$= \frac{1}{1+0} = 1.$$ (Can "jump" to that answer using shortcuts e.g. $\lim_{n\to\infty} \frac{n}{n+17} = \lim_{n\to\infty} \frac{n}{n} = 1$)

Since n^{th} term does NOT approach 0,
the series DIVERGES.

Question 2. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n}$$

$$= \sum \left(\frac{2^n}{4^n} + \frac{3^n}{4^n} \right) = \sum \frac{2^n}{4^n} + \sum \frac{3^n}{4^n}$$

$$= \underbrace{\sum \left(\frac{2}{4}\right)^n}_{\text{Geometric}} + \underbrace{\sum \left(\frac{3}{4}\right)^n}_{\text{Geometric}}$$

Geometric with $r = \frac{2}{4} = \frac{1}{2}$

Since $-1 < \frac{1}{2} < 1$, this series CONVERGES

Geometric with $r = \frac{3}{4}$

Since $-1 < \frac{3}{4} < 1$, this series CONVERGES

Original series is a sum of two convergent series and hence also CONVERGES

Question 3. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$$

Try integral test. $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^{3/2}}\,dx$

Sub $u = \ln x$ $\quad x = 2 \Rightarrow u = \ln 2$

$\downarrow$ $\quad x = \infty \Rightarrow u = \ln \infty = \infty$

$du = \frac{1}{x}\,dx$

$$\int_{x=2}^{x=\infty} \frac{1}{(\ln x)^{3/2}} \cdot \frac{1}{x}\,dx = \underbrace{\int_{u=\ln 2}^{u=\infty} \frac{1}{u^{3/2}}\,du}_{} = \int_{\ln 2}^{\infty} u^{-3/2}\,du$$

Shortcut: This CONVERGES because it's similar to a p-series with $p = \frac{3}{2} > 1$

$$= \left[\frac{u^{-1/2}}{-1/2}\right]_{\ln 2}^{\infty}$$

$$= \left[-\frac{2}{u^{1/2}}\right]_{\ln 2}^{\infty} = \left[\frac{2}{u^{1/2}}\right]_{\infty}^{\ln 2} = \frac{2}{(\ln 2)^{1/2}} - \underbrace{\frac{2}{\infty}}_{\to 0}$$

This is a FINITE number (∞ is in the DENOMINATOR)

i.e. the improper integral CONVERGES

So the series also converges by the integral test.

Question 4. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{5n + 10\sqrt{n}}$$

(Idea: When n is large then $5n + 10\sqrt{n} \approx 5n$)

Try limit comparison test. n^{th} term of given series is $a_n = \frac{1}{5n+10\sqrt{n}}$

Try $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{5n+10\sqrt{n}} \div \frac{1}{n} \right)$

$= \lim_{n\to\infty} \frac{n}{5n+10\sqrt{n}} = \lim_{n\to\infty} \frac{n}{5n+10\sqrt{n}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

SHORTCUT: $= \frac{1}{5}$ by looking at highest degree in top and bottom

$= \lim_{n\to\infty} \frac{1}{5 + \frac{10}{\sqrt{n}}}$

$= \frac{1}{5+0} = \frac{1}{5}$. Since this is not 0 and not ∞, we conclude $\sum a_n$ behaves the "same" as $\sum b_n$

We note that $\sum b_n = \sum \frac{1}{n}$ is the famous harmonic series, which DIVERGES.

Therefore by limit comparison, the original series also DIVERGES.

Question 5. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{5/4}}$$

We know $0 \le \cos^2 n \le 1$

Therefore $0 \le \dfrac{\cos^2 n}{n^{5/4}} \le \dfrac{1}{n^{5/4}}$

$$\sum \frac{\cos^2 n}{n^{5/4}} \le \underbrace{\sum \frac{1}{n^{5/4}}}_{\text{p-series with } p = \frac{5}{4} > 1 \text{ CONVERGES}}$$

Original series converges by basic comparison.

Question 6. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

METHOD 1. Ratio test. $a_n = \dfrac{1}{n2^n}$ $\quad a_{n+1} = \dfrac{1}{(n+1)2^{n+1}}$

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)2^{n+1}} \div \frac{1}{n2^n} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)2^{n+1}} \cdot \frac{n2^n}{1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \cdot \frac{1}{2} \right) = 1 \cdot \frac{1}{2} = \frac{1}{2} < 1$$

Since $R < 1$, the series CONVERGES.

METHOD 2. Note that $n2^n \geq 2^n$, so $\dfrac{1}{n2^n} \leq \dfrac{1}{2^n}$,

so $$\sum \frac{1}{n2^n} \leq \underbrace{\sum \frac{1}{2^n}}_{\text{convergent geometric series with } r = \frac{1}{2}}$$

Original series CONVERGES by basic comparison.

Question 7. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}$$

Try ratio test. $a_n = \frac{3^n}{n!}$ $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{3}{1} \cdot \frac{1}{n+1} \right| = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Since $R < 1$, the series CONVERGES.

Question 8. Is the series absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n}$$

The series is alternating. It converges "as is" since $\frac{1}{n\ln n} \to 0$.

What if we make all terms positive? We get $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

Try integral test. $\int_2^{\infty}\frac{1}{x\ln x}\,dx$ Sub $u=\ln x$ $\downarrow$ $du=\frac{1}{x}\,dx$ $x=2 \Rightarrow u=\ln 2$ $x=\infty \Rightarrow u=\ln\infty = \infty$

$$\int_{x=2}^{x=\infty}\frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int_{u=\ln 2}^{u=\infty}\frac{1}{u}\,du = \Big[\ln u\Big]_{u=\ln 2}^{u=\infty}$$

$= \underbrace{\ln\infty}_{\infty} - \ln\ln 2$ which DIVERGES.

So $\sum\frac{1}{n\ln n}$ diverges

So the original series CONVERGES CONDITIONALLY.

Question 9. Find the interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 \cdot 3^n}$$

Use ratio test. $a_n = \dfrac{(x-1)^n}{n^3 \cdot 3^n}$ $\quad a_{n+1} = \dfrac{(x-1)^{n+1}}{(n+1)^3 \cdot 3^{n+1}}$

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^3 \cdot 3^{n+1}} \div \frac{(x-1)^n}{n^3 \cdot 3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^3 \cdot 3^{n+1}} \cdot \frac{n^3 \cdot 3^n}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{(x-1)^{n+1}}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{3} \cdot \frac{n^3}{(n+1)^3} \cdot (x-1) \right|$$

$$= \lim_{n\to\infty} \left(\frac{1}{3} \cdot \frac{n^3}{(n+1)^3} \cdot |x-1| \right)$$

$$= \frac{1}{3} \cdot |x-1| \cdot \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \frac{1}{3} \cdot |x-1| \cdot 1 = \frac{|x-1|}{3}$$

Converges if $|x-1| < 3$, diverges if $|x-1| > 3$

$$-3 < x-1 < 3 \Rightarrow -2 < x < 4$$

$x=4$? $\sum \frac{(4-1)^n}{n^3 3^n} = \sum \frac{1}{n^3}$ converges. $x=-2$? $\sum \frac{(-2-1)^n}{n^3 \cdot 3^n} = \sum \frac{(-1)^n}{n^3}$ converges