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MAC 2312 Section U03 Test 3
Friday April 7th
Total possible score: 20 points (2 points per page)

Question 1. Evaluate the integral.

$$\int \sqrt{\tan x} \sec^4 x \, dx$$

$$\int \sqrt{\tan x} \underbrace{\sec^2 x}_{\substack{\text{even power} \\ \text{Easy to rewrite}}} \underbrace{\sec^2 x \, dx}_{\text{candidate for } du}$$

Substitute $u = \tan x \Rightarrow du = \sec^2 x \, dx$

$$\text{Integral} = \int \sqrt{\tan x} \left(1 + \tan^2 x\right) \sec^2 x \, dx$$

$$= \int \sqrt{u} \left(1 + u^2\right) du$$

$$= \int u^{1/2}\left(1 + u^2\right) du = \int \left(u^{1/2} + u^{5/2}\right) du$$

$$= \frac{u^{3/2}}{3/2} + \frac{u^{7/2}}{7/2} + C = \frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + C$$

$$= \frac{2}{3} \tan^{3/2} x + \frac{2}{7} \tan^{7/2} x + C$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	undef

Question 2. Evaluate the integral.

$$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2-1}}{x}\,dx$$

Looks like trig substitution

$x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta\,d\theta$

If $x = \sqrt{2}$ then $\sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

If $x = 2$ then $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\int_{x=\sqrt{2}}^{x=2} \frac{\sqrt{x^2-1}}{x}\,dx = \int_{\theta=\pi/4}^{\theta=\pi/3} \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \sqrt{\tan^2\theta}\,\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \tan\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \tan^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} (\sec^2\theta - 1)\,d\theta$$

$$= \Big[\tan\theta - \theta\Big]_{\pi/4}^{\pi/3} = \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$= \sqrt{3} - \frac{\pi}{3} - \left(1 - \frac{\pi}{4}\right) = \sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}$$

$$= \sqrt{3} - 1 - \frac{\pi}{12}$$

Question 3. Evaluate the integral.

$$\int_1^{\sqrt{3}} \frac{1}{(\sqrt{1+x^2})^3}\,dx$$

Looks like trig substitution

$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

If $x = 1$ then $\tan\theta = 1 \Rightarrow \theta = \pi/4$

If $x = \sqrt{3}$ then $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$$\int_{x=1}^{x=\sqrt{3}} \frac{1}{(\sqrt{1+x^2})^3}\,dx = \int_{\theta=\pi/4}^{\theta=\pi/3} \frac{1}{(\sqrt{1+\tan^2\theta})^3} \sec^2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{(\sqrt{\sec^2\theta})^3} \sec^2\theta\, d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta} \sec^2\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta}\, d\theta = \int_{\pi/4}^{\pi/3} \cos\theta\, d\theta = \Big[\sin\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{2}$$

Question 4. Evaluate the integral.

$$\int_8^{14} \frac{5x-17}{x^2-8x+7}\,dx$$

Looks like partial fractions. $x^2-8x+7=(x-1)(x-7)$

$$\frac{5x-17}{x^2-8x+7} = \frac{5x-17}{(x-1)(x-7)} = \frac{A}{x-1} + \frac{B}{x-7}$$

multiply both sides by $(x-1)(x-7)$

$$5x-17 = A(x-7)+B(x-1)$$
$$=(A+B)x+(-7A-B)$$

$$\Rightarrow \begin{aligned} A+B &= 5 \\ -7A-B &= -17 \\ \hline -6A &= -12 \end{aligned} \Rightarrow A=2 \Rightarrow B=3$$

$$\int_8^{14} \frac{5x-17}{x^2-8x+7}\,dx = \int_8^{14} \frac{5x-17}{(x-1)(x-7)}\,dx = \int_8^{14}\left(\frac{2}{x-1}+\frac{3}{x-7}\right)dx$$

$$= \Big[\,2\ln|x-1| + 3\ln|x-7|\,\Big]_8^{14}$$

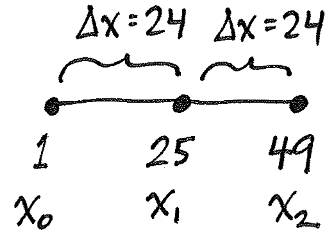
$$= \left(2\ln|14-1| + 3\ln|14-7|\right) - \left(2\ln|8-1| + 3\underbrace{\ln|8-7|}_{\ln 1\,=\,0}\right)$$

$$= 2\ln 13 + 3\ln 7 - 2\ln 7 \qquad = 2\ln 13 + \ln 7$$

Question 5. Use the Trapezoid Rule and Simpson's Rule to estimate the integral, using $n = 2$ subintervals.

$$\int_1^{49} \frac{1}{\sqrt{x}+1}\,dx$$

$$f(x) = \frac{1}{\sqrt{x}+1} \qquad [a,b] = [1,49] \qquad n=2$$



$$\Delta x = \frac{b-a}{n} = \frac{49-1}{2} = \frac{48}{2} = 24$$

$$f(x_0) = f(1) = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$f(x_1) = f(25) = \frac{1}{\sqrt{25}+1} = \frac{1}{5+1} = \frac{1}{6}$$

$$f(x_2) = f(49) = \frac{1}{\sqrt{49}+1} = \frac{1}{7+1} = \frac{1}{8}$$

Trapezoid approximation: $\frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + f(x_2)\right)$

$$= \frac{1}{2}\cdot 24\cdot\left(\frac{1}{2} + 2\cdot\frac{1}{6} + \frac{1}{8}\right) = 12\cdot\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right) = 6 + 4 + \frac{12}{8}$$

$$= 10 + \frac{3}{2} = \frac{23}{2}$$

Simpson's approximation: $\frac{1}{3}\Delta x\left(f(x_0) + 4f(x_1) + f(x_2)\right)$

$$= \frac{1}{3}\cdot 24\cdot\left(\frac{1}{2} + 4\cdot\frac{1}{6} + \frac{1}{8}\right) = 8\cdot\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{8}\right) = 4 + \frac{16}{3} + 1$$

$$= 5 + \frac{16}{3} = \frac{31}{3}$$

Question 6. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_0^\infty 2xe^{-x^2}\,dx$$

Consider $\int_0^M 2x e^{-x^2} dx$. (Let $M \to \infty$ at the <u>end</u>.)

Looks like substitution. Try $u = -x^2 \Rightarrow du = -2x\,dx$

$-du = 2x\,dx$

If $x=0$, then $u=0$. If $x=M$, then $u=-M^2$.

$$\int_{x=0}^{x=M} e^{-x^2}\cdot 2x\,dx = \int_{u=0}^{u=-M^2} e^u\cdot(-du)$$

$$= -\int_0^{-M^2} e^u\,du = \int_{-M^2}^0 e^u\,du = \Big[e^u\Big]_{u=-M^2}^{u=0}$$

$$= e^0 - e^{-M^2} = 1 - \frac{1}{e^{M^2}}$$

Then $\lim\limits_{M\to\infty}\left(1 - \frac{1}{e^{M^2}}\right) = 1 - 0 = \boxed{1}$

$\frac{1}{\infty}$ form

<u>CONVERGES</u>

Question 7. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_4^5 \frac{1}{(x-4)^{1/3}}\,dx$$

Improper because $x=4$ is the "problem" point.

Consider $\displaystyle\int_M^5 \frac{1}{(x-4)^{1/3}}\,dx$. At the end, let $M \to 4$ (from the right)

$$= \int_M^5 (x-4)^{-1/3}\,dx$$

Could substitute $u = x-4$ but it's the "easy" substitution where $du = dx$.

Shortcut: $\displaystyle\int (x-4)^n\,dx = \frac{(x-4)^{n+1}}{n+1}$

$$= \left[\frac{(x-4)^{2/3}}{2/3}\right]_M^5$$

$$= \frac{3}{2}\left[(x-4)^{2/3}\right]_M^5 = \frac{3}{2}\left(\underbrace{(5-4)^{2/3}}_{1} - (M-4)^{2/3}\right)$$

$$= \frac{3}{2}\left(1-(M-4)^{2/3}\right).$$ Then $$\lim_{M\to 4^+} \frac{3}{2}\left(1-(M-4)^{2/3}\right)$$

$$= \frac{3}{2}(1-0) = \boxed{\frac{3}{2}}$$

CONVERGES

Question 8. Find a formula for the general term of the sequence, starting with $n = 1$.

$$\frac{1}{2}, \frac{4}{5}, \frac{7}{8}, \frac{10}{11}, \ldots$$

If $n = 1$ then $a_n = \dfrac{1}{2}$

If $n = 2$ then $a_n = \dfrac{4}{5}$

If $n = 3$ then $a_n = \dfrac{7}{8}$

Numerators must be $1, 4, 7, 10, \ldots$

Denominators must be $2, 5, 8, 11, \ldots$

The formula $3n$ gives $3, 6, 9, 12, \ldots$

$3n-2$ gives $1, 4, 7, 10, \ldots$

$3n-1$ gives $2, 5, 8, 11, \ldots$

$$\boxed{a_n = \frac{3n-2}{3n-1}}$$

$$\frac{3\cdot 1 - 2}{3\cdot 1 - 1}, \frac{3\cdot 2 - 2}{3\cdot 2 - 1}, \frac{3\cdot 3 - 2}{3\cdot 3 - 1}, \ldots$$

Question 9. Determine whether the series converges or diverges. Explain.

$$\sum_{k=0}^{\infty} \frac{7k+1}{7k+2}$$

Note $\lim\limits_{k\to\infty} \dfrac{7k+1}{7k+2} = \lim\limits_{k\to\infty} \dfrac{7k+1}{7k+2} \cdot \dfrac{\frac{1}{k}}{\frac{1}{k}}$

$= \lim\limits_{k\to\infty} \dfrac{7+\frac{1}{k}}{7+\frac{2}{k}} = \dfrac{7+0}{7+0} = \dfrac{7}{7} = 1 \neq 0$

Since the k^{th} term does NOT approach 0, the series DIVERGES by the "pre-test"

Question 10. Determine whether the series converges or diverges. Explain.

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$$

$f(k) = \dfrac{1}{k(\ln k)^2}$ is a positive decreasing function

so we can use the integral test.

Check whether $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges or diverges.

Consider $\displaystyle\int_2^{M} \frac{1}{x(\ln x)^2}\,dx$. How to evaluate?

Try substitution. $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_{x=2}^{x=M} \underbrace{\frac{1}{(\ln x)^2}}_{1/u^2} \cdot \underbrace{\frac{1}{x}\,dx}_{du} = \int_{u=\ln 2}^{u=\ln M} \frac{1}{u^2}\,du$$

$$= \int_{\ln 2}^{\ln M} u^{-2}\,du = \left[\frac{u^{-1}}{-1}\right]_{\ln 2}^{\ln M} = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln M}$$

$$= \left[\frac{1}{u}\right]_{\ln M}^{\ln 2} = \frac{1}{\ln 2} - \frac{1}{\ln M}.$$ Then $\displaystyle\lim_{M\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln M}\right)$

$= \dfrac{1}{\ln 2} - 0$ CONVERGES