MAC2312 Section U03
Suggested problems for Test 3
(Test 3 is Friday April 7th, in class)

Idris Mercer

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1. Evaluate the integral.

$$\int_0^{\pi/2} \cos^3 x \, dx$$

Guidelines: Consider candidates for du. Even powers are easy to rewrite.

Try $u = \sin x \Rightarrow du = \cos x \, dx$

If $x = 0$, then $u = \sin 0 = 0$

If $x = \pi/2$, then $u = \sin\frac{\pi}{2} = 1$

$$\text{Integral} = \int_{x=0}^{x=\pi/2} \cos^2 x \cdot \cos x \, dx = \int_{x=0}^{x=\pi/2} \underbrace{(1-\sin^2 x)}_{1-u^2} \underbrace{\cos x \, dx}_{du}$$

$$= \int_{u=0}^{u=1} (1-u^2) \, du = \left[u - \frac{u^3}{3} \right]_{u=0}^{u=1}$$

$$= \left(1 - \frac{1}{3}\right) - (0 - 0)$$

$$= \frac{2}{3}$$

2. Evaluate the integral.

$$\int_0^{\pi/4} \sin^3 x \cos^3 x \, dx$$

$$\int_{x=0}^{x=\pi/4} \sin^3 x \underbrace{\cos^2 x}_{\text{easy to rewrite}} \underbrace{\cos x \, dx}_{\text{candidate for } du}$$

Try $u = \sin x$

$\Rightarrow du = \cos x \, dx$

If $x=0$, then $u = \sin 0 = 0$

If $x = \frac{\pi}{4}$, then $u = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\int_{x=0}^{x=\pi/4} \sin^3 x \left(1-\sin^2 x\right) \underbrace{\cos x \, dx}_{du}$$

$$= \int_{u=0}^{u=1/\sqrt{2}} u^3 \left(1-u^2\right) du = \int_0^{1/\sqrt{2}} \left(u^3 - u^5\right) du$$

$$= \left[\frac{u^4}{4} - \frac{u^6}{6} \right]_0^{1/\sqrt{2}} = \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4 - \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 - 0$$

$$= \frac{1}{4}\cdot\frac{1}{2^2} - \frac{1}{6}\cdot\frac{1}{2^3} = \frac{1}{4\cdot 4} - \frac{1}{6\cdot 8} = \frac{1}{16} - \frac{1}{48}$$

$$= \frac{3}{48} - \frac{1}{48} = \frac{2}{48} = \frac{1}{24}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\sin^2 x + \cos^2 x = 1$$

divide by $\cos^2 x$

$$\tan^2 x + 1 = \sec^2 x$$

3. Evaluate the integral.

$$\int \sec^5 x \tan^3 x \, dx$$

Again, think about candidates for du, and remember that even powers are easy to rewrite.

$$\text{Integral} = \int \underbrace{\sec^4 x \tan^2 x}_{\text{easy to rewrite if needed}} \cdot \underbrace{\sec x \tan x \, dx}_{\text{Candidate for du}}$$

So maybe try $u = \sec x$

$$u = \sec x$$

$$\Rightarrow du = \sec x \tan x \, dx$$

$$\text{Integral} = \int \sec^4 x \left(\sec^2 x - 1\right) \sec x \tan x \, dx = \int u^4 (u^2 - 1) \, du$$

$$= \int (u^6 - u^4) \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

4. Evaluate the integral.

$$\int x\sqrt{4-x^2}\,dx$$

METHOD 1: Trig substitution. $x = 2\sin\theta \Rightarrow x^2 = 4\sin^2\theta$

$\Rightarrow dx = 2\cos\theta\, d\theta$

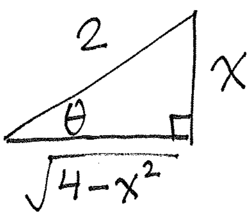
$$\int x\sqrt{4-x^2}\,dx = \int 2\sin\theta\sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$= \int 2\sin\theta\sqrt{4\cos^2\theta}\; 2\cos\theta\, d\theta = \int 2\sin\theta\; 2\cos\theta\; 2\cos\theta\, d\theta$$

$$= 8\int \cos^2\theta \sin\theta\, d\theta$$ Then $u = \cos\theta$, $du = -\sin\theta\, d\theta$, $-du = \sin\theta\, d\theta$ $\quad 8\int u^2(-du)$

$$= -8\int u^2\, du = -\frac{8}{3}u^3 + C = -\frac{8}{3}\cos^3\theta + C$$

Then rewrite using x. Recall $x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} = \frac{OPP}{HYP}$



$\cos\theta = \frac{ADJ}{HYP} = \frac{\sqrt{4-x^2}}{2}$ Answer: $\boxed{-\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C}$

METHOD 2: U-substitution. $u = 4 - x^2 \Rightarrow du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$\text{Integral} = \int \sqrt{4-x^2}\; x\,dx = \int \sqrt{u}\cdot \frac{-1}{2}du = -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\frac{u^{3/2}}{3/2} = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} = -\frac{1}{3}u^{3/2} = \boxed{-\frac{1}{3}(4-x^2)^{3/2} + C}$$

Note that the two answers are the same because $\left(\frac{\sqrt{4-x^2}}{2}\right)^3 = \frac{(4-x^2)^{3/2}}{8}$

Remember $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 - \sin^2\theta = \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

5. Evaluate the integral.

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$$

Looks like trig sub. Try $x = 3\sin\theta \Rightarrow x^2 = 9\sin^2\theta$

$\Rightarrow dx = 3\cos\theta\, d\theta$

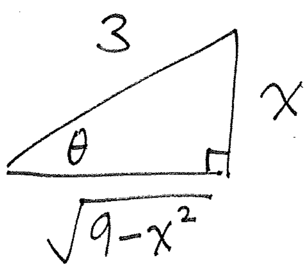
$$\text{Integral} = \int \frac{1}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}\,3\cos\theta\, d\theta = \int \frac{1}{9\sin^2\theta\sqrt{9\cos^2\theta}}\,3\cos\theta\, d\theta$$

$$= \int \frac{1}{9\sin^2\theta\, 3\cos\theta}\,3\cos\theta\, d\theta = \int \frac{1}{9\sin^2\theta}\,d\theta = \frac{1}{9}\int \csc^2\theta\, d\theta$$

Now what? Similar strategy as $\int \sec^2\theta\, d\theta$. Recall $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

Also $\frac{d}{d\theta}(\cot\theta) = -\csc^2\theta$. So $\frac{1}{9}\int \csc^2\theta\, d\theta = -\frac{1}{9}\cot\theta$

Then rewrite using x. Recall $x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3} = \frac{\text{OPP}}{\text{HYP}}$



$\tan\theta = \frac{\text{OPP}}{\text{ADJ}}$

$\cot\theta = \frac{\text{ADJ}}{\text{OPP}} = \frac{\sqrt{9-x^2}}{x}$

Answer: $-\frac{1}{9}\cdot\frac{\sqrt{9-x^2}}{x} + C$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$
0	0	1	0	undef
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$
$\pi/2$	1	0	undef	0

6. Evaluate the integral.

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

$\Rightarrow x^2 = 4\sin^2\theta$

Looks like trig sub. Try $x = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta\, d\theta$

If $x=1$ then $1 = 2\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

If $x=\sqrt{3}$ then $\sqrt{3} = 2\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\text{Integral} = \int_{\theta=\pi/6}^{\theta=\pi/3} \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{4\sin^2\theta\sqrt{4\cos^2\theta}}\, 2\cos\theta\, d\theta = \int_{\pi/6}^{\pi/3} \frac{1}{4\sin^2\theta\, 2\cos\theta}\, 2\cos\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{4\sin^2\theta}\, d\theta = \frac{1}{4}\int_{\pi/6}^{\pi/3} \csc^2\theta\, d\theta = \left[-\frac{1}{4}\cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{4}\left[\cot\theta\right]_{\pi/3}^{\pi/6} = \frac{1}{4}\left(\cot\frac{\pi}{6} - \cot\frac{\pi}{3}\right)$$

$$= \frac{1}{4}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{1}{4}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{1}{4}\cdot\frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{6}$$

or $\frac{1}{2\sqrt{3}}$

$\sin^2\theta + \cos^2\theta = 1$ ↓ divide by $\cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

7. Evaluate the integral.

$$\int \frac{x}{\sqrt{9+x^2}}\,dx$$

METHOD 1: Trig sub. Try $x = 3\tan\theta$ $\Rightarrow x^2 = 9\tan^2\theta$

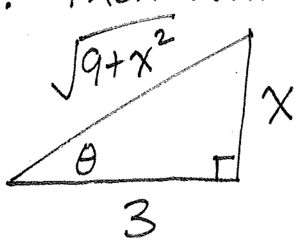
$\Rightarrow dx = 3\sec^2\theta\,d\theta$

$$\text{Integral} = \int \frac{3\tan\theta}{\sqrt{9+9\tan^2\theta}}\,3\sec^2\theta\,d\theta = \int \frac{3\tan\theta}{\sqrt{9\sec^2\theta}}\,3\sec^2\theta\,d\theta$$

$$= \int \frac{3\tan\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta = 3\int \sec\theta\tan\theta\,d\theta$$ KNOWN derivative!

$= 3\sec\theta + C$. Then rewrite using x. Recall $x = 3\tan\theta$.

$\frac{x}{3} = \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$



$\cos\theta = \frac{\text{ADJ}}{\text{HYP}}$ $\sec\theta = \frac{\text{HYP}}{\text{ADJ}}$

$\sec\theta = \frac{\sqrt{9+x^2}}{3}$

Answer: $3\cdot\frac{\sqrt{9+x^2}}{3} + C = \sqrt{9+x^2} + C.$

METHOD 2: U-sub. Try $u = 9 + x^2 \Rightarrow du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\text{Integral} = \int \frac{1}{\sqrt{9+x^2}}\,x\,dx = \int \frac{1}{\sqrt{u}}\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\,\frac{u^{1/2}}{1/2} + C = u^{1/2} + C = \sqrt{9+x^2} + C.$$

8. Evaluate the integral.

$$\int \frac{x+6}{x^2-3x-4}\,dx$$

Looks like partial fractions. $x^2-3x-4=(x+1)(x-4)$

$$\frac{x+6}{x^2-3x-4} = \frac{x+6}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

multiply both sides by $(x+1)(x-4)$

$$\begin{aligned} x+6 &= A(x-4) + B(x+1) \\ &= Ax - 4A + Bx + B \\ &= (A+B)x + (-4A+B) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} 1 &= A+B \\ 6 &= -4A+B \end{aligned}\right\} \Rightarrow \begin{aligned} 1-6 &= (A+B)-(-4A+B) \\ -5 &= A+B+4A-B = 5A \end{aligned} \quad \begin{aligned} &\Rightarrow A=-1 \\ &\Rightarrow B = 2 \end{aligned}$$

So $$\frac{x+6}{(x+1)(x-4)} = \frac{-1}{x+1} + \frac{2}{x-4}$$

So $$\int \frac{x+6}{(x+1)(x-4)}\,dx = \int\left(\frac{-1}{x+1} + \frac{2}{x-4}\right)dx$$

$$= -1\ln|x+1| + 2\ln|x-4| + C$$

9. Evaluate the integral.

$$\int_2^8 \frac{5x+13}{x^2+4x-5}\,dx$$

Looks like partial fractions. $x^2+4x-5=(x+5)(x-1)$

$$\frac{5x+13}{x^2+4x-5}=\frac{5x+13}{(x+5)(x-1)}=\frac{A}{x+5}+\frac{B}{x-1}$$

mult. by $(x+5)(x-1)$

$$5x+13 = A(x-1)+B(x+5)$$
$$= (A+B)x+(-A+5B)$$

$$\Rightarrow \left.\begin{aligned} 5&=A+B\\ 13&=-A+5B\end{aligned}\right\} \Rightarrow 5+13=(A+B)+(-A+5B)=6B \Rightarrow 18=6B$$
$$\Rightarrow B=3$$
$$\Rightarrow A=2$$

$$\int_2^8 \frac{5x+13}{(x+5)(x-1)}\,dx=\int_2^8\left(\frac{2}{x+5}+\frac{3}{x-1}\right)dx$$

$$=\Big[2\ln|x+5|+3\ln|x-1|\Big]_2^8$$

$$=\big(2\ln|8+5|+3\ln|8-1|\big)-\big(2\ln|2+5|+3\ln|2-1|\big)$$

$$= 2\ln 13+3\ln 7-2\ln 7-\underbrace{3\ln 1}_{0}$$

$$=\boxed{2\ln 13+\ln 7}$$

10. Evaluate the integral.

$$\int \frac{2x^2 - 9x - 9}{x^3 - 9x}\, dx$$

Looks like partial fractions. $x^3 - 9x = x(x^2-9) = x(x+3)(x-3)$

$$\frac{2x^2-9x-9}{x(x+3)(x-3)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-3}$$

multiply both sides by $x(x+3)(x-3)$

$$\begin{aligned} 2x^2-9x-9 &= A(x+3)(x-3) + Bx(x-3) + Cx(x+3) \\ &= A(x^2-9) + B(x^2-3x) + C(x^2+3x) \\ &= (A+B+C)x^2 + (-3B+3C)x + (-9A) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} 2 &= A+B+C \\ -9 &= -3B+3C \\ -9 &= -9A \end{aligned}\right\} \Rightarrow A = 1 \Rightarrow B+C = 1 \Rightarrow 3B+3C = 3$$

$$\begin{aligned} 3B+3C &= 3 \\ -3B+3C &= -9 \\ \hline 6C &= -6 \end{aligned} \quad \Rightarrow C = -1 \quad \Rightarrow B = 2$$

$$\int \frac{2x^2-9x-9}{x(x+3)(x-3)}\, dx = \int \left(\frac{1}{x} + \frac{2}{x+3} + \frac{-1}{x-3} \right) dx$$

$$= \ln|x| + 2\ln|x+3| - \ln|x-3| + C$$

11. Evaluate the integral.

$$\int_2^3 \frac{1}{x^3 - x}\,dx$$

Partial fractions. $x^3 - x = x(x^2-1) = x(x+1)(x-1)$

$$\frac{1}{x^3-x} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$\downarrow$ multiply both sides by $x(x+1)(x-1)$

$$\begin{aligned} 1 &= A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \\ &= A(x^2-1) + B(x^2-x) + C(x^2+x) \\ &= (A+B+C)x^2 + (-B+C)x + (-A) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} 0 &= A+B+C \\ 0 &= -B+C \\ 1 &= -A \end{aligned}\right\} \Rightarrow A = -1 \Rightarrow \begin{aligned} B+C &= 1 \\ -B+C &= 0 \\ \hline 2C &= 1 \end{aligned} \Rightarrow C = \frac{1}{2} \Rightarrow B = \frac{1}{2}$$

$$\int_2^3 \frac{1}{x^3-x}\,dx = \int_2^3 \left(\frac{-1}{x} + \frac{1/2}{x+1} + \frac{1/2}{x-1} \right) dx$$

$$= \left[-\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| \right]_2^3$$

$$= \left(-\ln 3 + \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2\right) - \left(-\ln 2 + \frac{1}{2}\ln 3 + \frac{1}{2}\underbrace{\ln 1}_{0}\right)$$

$$= -\ln 3 + \frac{1}{2}\cdot 2\ln 2 + \frac{1}{2}\ln 2 + \ln 2 - \frac{1}{2}\ln 3$$

$$= \frac{5}{2}\ln 2 - \frac{3}{2}\ln 3$$

12. Evaluate the integral.

$$\int \frac{2x^2+3}{x^3-2x^2+x}\,dx$$

Partial fractions. $x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$

$$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

repeated factor

multiply both sides by $x(x-1)^2$

$$2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$$
$$= A(x^2-2x+1) + B(x^2-x) + Cx$$
$$= (A+B)x^2 + (-2A-B+C)x + A$$

$$\Rightarrow \left.\begin{aligned} 2 &= A+B \\ 0 &= -2A-B+C \\ 3 &= A \end{aligned}\right\} \Rightarrow \begin{aligned} &A=3 \Rightarrow B=-1 \\ &\Rightarrow 0=-6+1+C \Rightarrow C=5 \end{aligned}$$

$$\int \frac{2x^2+3}{x(x-1)^2}\,dx = \int\left(\frac{3}{x} + \frac{-1}{x-1} + \frac{5}{(x-1)^2}\right)dx$$

$$= 3\int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + 5\int (x-1)^{-2}\,dx$$

NOTHING TO DO WITH LOGARITHMS

$$= 3\ln|x| - \ln|x-1| + 5\cdot\frac{(x-1)^{-1}}{-1} + C$$

$$= 3\ln|x| - \ln|x-1| - \frac{5}{x-1} + C$$

13. Evaluate the integral.

$$\int \frac{2x+7}{x^2+1}\,dx$$

x^2+1 does NOT factor so this is NOT partial fractions.

$$\int \frac{2x+7}{x^2+1}\,dx = \int \left(\frac{2x}{x^2+1} + \frac{7}{x^2+1} \right) dx$$

$$= \underbrace{\int \frac{2x}{x^2+1}\,dx}_{\text{Try sub. } u = x^2+1 \;\Rightarrow du = 2x\,dx} + \int \frac{7}{x^2+1}\,dx$$

$$= \int \frac{1}{u}\,du + 7\int \frac{1}{x^2+1}\,dx$$

$$= \ln|u| + 7\arctan x + C$$

$$= \ln|x^2+1| + 7\arctan x + C$$

$$= \ln(x^2+1) + 7\arctan x + C$$

$|x^2+1| = x^2+1$ since x^2+1 is positive

By the way: From Question 11, we know the exact answer is $\frac{5}{2}\ln 2 - \frac{3}{2}\ln 3$ which as a decimal is about 0.0849

14. Use the Trapezoid Rule and Simpson's rule to estimate the integral, using $n = 2$ subintervals.

$$\int_2^3 \frac{1}{x^3 - x}\, dx$$

$[a,b] = [2,3]$ $\quad n=2 \quad \Delta x = \frac{b-a}{n} = \frac{3-2}{2} = \frac{1}{2}$ or 0.5

Three points: $x_0 = 2$, $x_1 = 2.5 = \frac{5}{2}$, $x_2 = 3$

Trapezoid estimate: $\frac{1}{2} \cdot \Delta x \cdot \left(f(x_0) + 2f(x_1) + f(x_2) \right)$

Simpson's estimate: $\frac{1}{3} \cdot \Delta x \cdot \left(f(x_0) + 4f(x_1) + f(x_2) \right)$

$$f(x_0) = f(2) = \frac{1}{2^3 - 2} = \frac{1}{8-2} = \frac{1}{6}$$

$$f(x_1) = f\left(\frac{5}{2}\right) = \frac{1}{\frac{125}{8} - \frac{5}{2}} \cdot \frac{8}{8} = \frac{8}{125-20} = \frac{8}{105}$$

$$f(x_2) = f(3) = \frac{1}{3^3 - 3} = \frac{1}{27-3} = \frac{1}{24}$$

Trapezoid estimate: $\frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{1}{6} + 2 \cdot \frac{8}{105} + \frac{1}{24} \right)$

Simpson's estimate: $\frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{6} + 4 \cdot \frac{8}{105} + \frac{1}{24} \right)$

On a test, can leave your answers in that form.

If you're curious: $\frac{1}{4}\left(\frac{1}{6} + \frac{16}{105} + \frac{1}{24} \right) = \frac{101}{1120} \approx 0.0902$

$\frac{1}{6}\left(\frac{1}{6} + \frac{32}{105} + \frac{1}{24} \right) = \frac{431}{5040} \approx 0.0855$

15. Evaluate the integral.

$$\int \frac{1}{x^3+x}\,dx$$

Partial fractions: $x^3+x = x(x^2+1)$

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Degree of top should be one less than degree of bottom

multiply both sides by $x(x^2+1)$

$$1 = A(x^2+1)+(Bx+C)x$$
$$= Ax^2+A+Bx^2+Cx$$

$$\Rightarrow \left.\begin{array}{l} 0 = A+B \\ 0 = C \\ 1 = A \end{array}\right\} \Rightarrow A=1,\ B=-1,\ C=0$$

$$\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{-x}{x^2+1}\right) dx$$

$$= \int \frac{1}{x}\,dx - \underbrace{\int \frac{x}{x^2+1}\,dx}_{\text{Sub } u = x^2+1}$$

Sub $u = x^2+1 \Rightarrow du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \ln|x| - \int \frac{1}{u}\cdot\frac{1}{2}\,du = \ln|x| - \frac{1}{2}\ln|u| + C$$
$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

By the way: Using Question 15, the exact answer is $\left[\ln|x| - \frac{1}{2}\ln(x^2+1)\right]_1^3 = \left(\ln 3 - \frac{1}{2}\ln 10\right) - \left(\underbrace{\ln 1}_{0} - \frac{1}{2}\ln 2\right)$

$= \ln 3 - \frac{1}{2}(\ln 2 + \ln 5) + \frac{1}{2}\ln 2 = \ln 3 - \frac{1}{2}\ln 5 \approx 0.294$

16. Use the Trapezoid Rule and Simpson's rule to estimate the integral, using $n = 4$ subintervals.

$$\int_1^3 \frac{1}{x^3+x}\,dx$$

$[a,b] = [1,3]$ $n=4$ $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$ or 0.5

$x_0 = 1$ $x_1 = 1.5 = \frac{3}{2}$ $x_2 = 2$ $x_3 = 2.5 = \frac{5}{2}$ $x_4 = 3$

$f(x_0) = f(1) = \frac{1}{1^3+1} = \frac{1}{1+1} = \frac{1}{2}$

$f(x_1) = f(\frac{3}{2}) = \frac{1}{\frac{27}{8}+\frac{3}{2}} \cdot \frac{8}{8} = \frac{8}{27+12} = \frac{8}{39}$

$f(x_2) = f(2) = \frac{1}{2^3+2} = \frac{1}{8+2} = \frac{1}{10}$

$f(x_3) = f(\frac{5}{2}) = \frac{1}{\frac{125}{8}+\frac{5}{2}} \cdot \frac{8}{8} = \frac{8}{125+20} = \frac{8}{145}$

$f(x_4) = f(3) = \frac{1}{3^3+3} = \frac{1}{27+3} = \frac{1}{30}$

Trapezoid estimate: $\frac{1}{2}\cdot\Delta x\cdot\left(f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right)$

$= \frac{1}{4}\left(\frac{1}{2} + \frac{16}{39} + \frac{1}{5} + \frac{16}{145} + \frac{1}{30}\right)$ ← On test, can stop there. This is $\frac{7091}{22620} \approx 0.313$

Simpson's estimate: $\frac{1}{3}\cdot\Delta x\cdot\left(f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right)$

$= \frac{1}{6}\left(\frac{1}{2} + \frac{32}{39} + \frac{1}{5} + \frac{32}{145} + \frac{1}{30}\right)$ ← On test, can stop there. This is $\frac{223}{754} \approx 0.296$

17. Evaluate the integral.

$$\int \frac{x+1}{x^3+x}\,dx$$

Partial fractions: $x^3+x = x(x^2+1)$

$$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Degree of top should be one less than degree of bottom

multiply both sides by $x(x^2+1)$

$$x+1 = A(x^2+1) + (Bx+C)x$$
$$= Ax^2 + A + Bx^2 + Cx$$

$$\Rightarrow \left.\begin{aligned} 0 &= A+B \\ 1 &= C \\ 1 &= A \end{aligned}\right\} \Rightarrow A=1,\ B=-1,\ C=1$$

$$\int \frac{x+1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} + \frac{-x+1}{x^2+1}\right)dx = \int\left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x}\,dx - \underbrace{\int \frac{x}{x^2+1}\,dx}_{\substack{\text{Sub } u=x^2+1 \\ du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx}} + \int \frac{1}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan x + C$$

18. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Consider $\displaystyle\int_1^M \frac{1}{x^2}\,dx = \int_1^M x^{-2}\,dx$

$\displaystyle = \left[\frac{x^{-1}}{-1}\right]_1^M = \left[-\frac{1}{x}\right]_1^M = \left[\frac{1}{x}\right]_M^1$

$\displaystyle = 1 - \frac{1}{M}$

Then $\displaystyle\lim_{M\to\infty}\left(1 - \frac{1}{M}\right) = 1 - 0 = 1$

The integral <u>converges</u> and its value is 1

19. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_0^\infty e^{-x}\,dx$$

Consider $\int_0^M e^{-x}\,dx$

Sub $u = -x \Rightarrow du = -1dx \Rightarrow -1du = dx$

If $x=0$, then $u=0$

If $x=M$, then $u=-M$

$$\int_{x=0}^{x=M} e^{-x}\,dx = \int_{u=0}^{u=-M} e^{u}\cdot -1\,du$$

$$= -\int_0^{-M} e^u\,du = \int_{-M}^{0} e^u\,du = \left[e^u\right]_{-M}^{0}$$

$$= e^0 - e^{-M} = 1 - e^{-M} = 1 - \frac{1}{e^M}$$

Then $\lim_{M\to\infty}\left(1 - \frac{1}{e^M}\right) = 1 - 0 = 1$

Integral converges and its value is 1

NOTE: Possible shortcut not using substitution: $\int e^{-x}\,dx$

$= -e^{-x} + C$

20. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_0^\infty \frac{x}{1+x^2}\,dx$$

Consider $\displaystyle\int_0^M \frac{x}{1+x^2}\,dx.$

Substitute $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

If $x=0$, then $u=1$

If $x=M$, then $u=1+M^2$

$$\int_{x=0}^{x=M} \frac{1}{1+x^2}\,x\,dx = \int_{u=1}^{u=1+M^2} \frac{1}{u}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^{1+M^2} \frac{1}{u}\,du = \frac{1}{2}\Big[\ln|u|\Big]_1^{1+M^2}$$

$$= \frac{1}{2}\Big(\ln(1+M^2) - \underbrace{\ln 1}_{0}\Big) = \frac{1}{2}\ln(1+M^2)$$

Then $\displaystyle\lim_{M\to\infty} \frac{1}{2}\ln(1+M^2) = \infty$ $(\ln\infty)$

The improper integral DIVERGES

21. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

Consider $\displaystyle\int_0^M \frac{1}{1+x^2}\,dx$.

We know an antiderivative of $\dfrac{1}{1+x^2}$. NOTHING to do with logarithms!

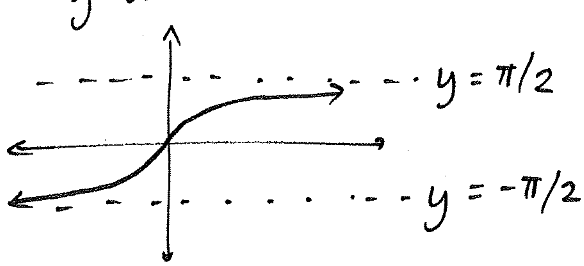
$$\int_0^M \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_0^M$$

$$= \arctan M - \underbrace{\arctan 0}_{0} = \arctan M.$$

Then $\displaystyle\lim_{M\to\infty} \arctan M = \frac{\pi}{2}$.

The improper integral CONVERGES and its value is $\dfrac{\pi}{2}$.

y = arctan x



22. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_3^4 \frac{1}{\sqrt{x-3}}\,dx$$

Note: $x = 3$ is the "problem" because the function $f(x) = \dfrac{1}{\sqrt{x-3}}$ becomes infinite there

Consider $\displaystyle\int_M^4 \frac{1}{\sqrt{x-3}}\,dx = \int_M^4 (x-3)^{-1/2}\,dx$

Substitute $u = x-3$ If $x = M$, then $u = M-3$

$du = 1\,dx$ If $x = 4$, then $u = 4-3 = 1$

$$\int_{x=M}^{x=4} (x-3)^{-1/2}\,dx = \int_{u=M-3}^{u=1} u^{-1/2}\,du$$

$$= \left[\frac{u^{1/2}}{1/2} \right]_{M-3}^{1} = \left[2u^{1/2} \right]_{M-3}^{1}$$

$$= 2\cdot 1 - 2(M-3)^{1/2} = 2 - 2\sqrt{M-3}$$

Then $\displaystyle\lim_{M\to 3^+} \left(2 - 2\sqrt{M-3}\right) = 2 - 2\sqrt{3-3}$

$$= 2 - 0 = 2$$

The improper integral CONVERGES and its value is 2

23. Determine whether the integral converges or diverges, and if it converges, find its value.

$$\int_3^4 \frac{1}{(x-3)^2}\,dx$$

$x=3$ is the "problem" point, so

consider $\displaystyle\int_M^4 \frac{1}{(x-3)^2}\,dx = \int_M^4 (x-3)^{-2}\,dx$

Sub $u = x-3$ $\quad$ If $x=M$, then $u = M-3$

$du = 1\,dx$ $\quad$ If $x=4$, then $u = 4-3 = 1$

$$\int_{x=M}^{x=4} (x-3)^{-2}\,dx = \int_{u=M-3}^{u=1} u^{-2}\,du$$

$$= \left[\frac{u^{-1}}{-1}\right]_{M-3}^{1} = \left[-\frac{1}{u}\right]_{M-3}^{1} = \left[\frac{1}{u}\right]_{1}^{M-3}$$

$$= \frac{1}{M-3} - 1$$

Then $\displaystyle\lim_{M\to 3^+}\left(\frac{1}{M-3} - 1\right) = +\infty$

The improper integral DIVERGES

24. Find a formula for the general term of the sequence, starting with $n = 1$.

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$$

$1 = 3^0 \quad 3 = 3^1 \quad 9 = 3^2 \quad 27 = 3^3$

The given sequence is

$$\frac{1}{3^0}, \frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots$$

$$\left\{ \frac{1}{3^{n-1}} \right\}_{n=1}^{\infty}$$ if starting with $n=1$

or $$a_n = \frac{1}{3^{n-1}}$$

25. Find a formula for the general term of the sequence, starting with $n = 1$.

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$$

$n=1$ must give $\frac{1}{2}$ $\quad a_1 = \frac{1}{2}$

$n=2$ must give $\frac{3}{4}$ $\quad a_2 = \frac{3}{4}$

$n=3$ must give $\frac{5}{6}$ $\quad a_3 = \frac{5}{6}$

$n=4$ must give $\frac{7}{8}$ $\quad a_4 = \frac{7}{8}$

$a_n = \frac{?}{?}$

$$a_n = \frac{2n-1}{2n} \quad \text{or} \quad 1 - \frac{1}{2n}$$

$$\left\{ \frac{2n-1}{2n} \right\}_{n=1}^{\infty}$$

26. Evaluate each of the following limits.

a. $\lim\limits_{n\to\infty} \frac{1}{2^n}$

b. $\lim\limits_{n\to\infty} \frac{n}{2^n}$

c. $\lim\limits_{n\to\infty} \frac{2n-1}{2n}$

d. $\lim\limits_{n\to\infty} \frac{\ln n}{n}$

e. $\lim\limits_{n\to\infty} \frac{\pi^n}{4^n}$

a. 0 because it has the form $\frac{1}{\infty}$

b. 0 because 2^n approaches infinity much faster than n approaches infinity. Can also use L'Hopital's rule.

c. 1 $\quad \frac{2n-1}{2n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{2-\frac{1}{n}}{2} \to \frac{2-0}{2}$ $=1$

d. 0 because $\ln n$ approaches infinity much slower than n approaches infinity. Can also use L'Hopital's rule.

e. 0 because $\frac{\pi^n}{4^n} = \left(\frac{\pi}{4}\right)^n$

and $\frac{\pi}{4}$ is a constant between -1 and 1

27. Determine whether the series converges. If it converges, find its value.

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$$

$$\left(\frac{2}{3}\right)^{3}+\left(\frac{2}{3}\right)^{4}+\left(\frac{2}{3}\right)^{5}+\left(\frac{2}{3}\right)^{6}+\cdots$$

Geometric series with $r=\frac{2}{3}$

and $a=\left(\frac{2}{3}\right)^{3}=\frac{8}{27}$.

Since r is between -1 and 1,
this geometric series converges.
The value of its sum is

$$\frac{a}{1-r}=\frac{\frac{8}{27}}{1-\frac{2}{3}}=\frac{\frac{8}{27}}{\frac{1}{3}}=\frac{8}{27}\cdot\frac{3}{1}$$

$$=\frac{8}{9}$$

28. Determine whether the series converges. If it converges, find its value.

$$\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1}$$

$$\left(-\frac{3}{4}\right)^0 + \left(-\frac{3}{4}\right)^1 + \left(-\frac{3}{4}\right)^2 + \dots$$

Geometric series with $r = -\frac{3}{4}$

and $a = \left(-\frac{3}{4}\right)^0 = 1$

Since r is between -1 and 1, this geometric series converges.

The value of its sum is

$$\frac{a}{1-r} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

29. Determine whether the series converges or diverges.

$$\sum_{k=3}^{\infty} \frac{1}{k-2}$$

$$\frac{1}{3-2} + \frac{1}{4-2} + \frac{1}{5-2} + \frac{1}{6-2} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This literally <u>is</u> THE HARMONIC SERIES which we know DIVERGES

30. Determine whether the series converges or diverges.

$$\sum_{k=5}^{\infty} \left(\frac{e}{\pi}\right)^{k-1}$$

This is a geometric series with $r = \frac{e}{\pi} \approx \frac{2.7}{3.1}$

Since r is between -1 and 1, this geometric series CONVERGES

31. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

This is a p-series with $p = \frac{1}{2}$

Since $p \leq 1$, this p-series DIVERGES.

(Also, you can use the integral test.)

32. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} k^{-2/3}$$

$$\sum_{k=1}^{\infty} k^{-2/3} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

This is a p-series with $p = \frac{2}{3}$

Since $p \leq 1$, this p-series DIVERGES.

(Also, you can use the integral test.)

33. Determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$$

Note that $\displaystyle \lim_{k\to\infty} \frac{k^2+1}{k^2+3}$

$$= \lim_{k\to\infty} \frac{k^2+1}{k^2+3} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{1+\frac{1}{k^2}}{1+\frac{3}{k^2}}$$

$$= \frac{1+0}{1+0} = 1$$ which is not 0

Since the k^{th} term of the series does NOT approach 0, the Divergence Pre-Test says the series must DIVERGE.

34. Determine whether the series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

Not a geometric series

Not a p-series

The Divergence Pre-Test doesn't help

We can use the INTEGRAL TEST.

$\int_2^\infty \frac{1}{x \ln x}\,dx = ?$ Consider $\int_2^M \frac{1}{x \ln x}\,dx$.

Try substituting $u = \ln x$. Then $du = \frac{1}{x}\,dx$.

If $x = 2$, then $u = \ln 2$. If $x = M$, then $u = \ln M$.

$$\int_{x=2}^{x=M} \frac{1}{\ln x} \cdot \frac{1}{x}\,dx = \int_{u=\ln 2}^{u=\ln M} \frac{1}{u}\,du$$

$$= \Big[\ln|u| \Big]_{\ln 2}^{\ln M} = \ln|\ln M| - \underbrace{\ln|\ln 2|}_{\text{just a number}}$$

$\lim_{M\to\infty} \ln|\ln M| = \infty$, so the integral diverges

By the integral test, the series also DIVERGES.