Calculus II is largely about **integrals**.

An integral is like a 'smoothed' sum of a very large number of very small pieces.

One of the most common uses of integrals is computing the **area under a curve**.

(Here, 'under a curve' really means 'between the curve and the x-axis', and 'area' really means 'net signed area'. We'll say more about this soon.)

The integral of a function $f(x)$ on an interval $[a, b]$ is written as follows.

$$\int_a^b f(x)\,dx$$

Why might we care about the area under a curve? Here's one reason:

FACT: The area under a *velocity* curve tells you the *displacement*. (Why is that?)

Riemann sums

The integral of $f(x)$ on the interval $[a, b]$ can be approximated in the following way.

We will divide $[a, b]$ into n equal subintervals and build a rectangle on each subinterval.

Then the width of each subinterval is

$$\Delta x = \frac{b - a}{n}$$

and the endpoints of the subintervals are $n+1$ different points, which we call the 'grid points'

$$\begin{aligned} x_0 &= a, \\ x_1 &= a + \Delta x, \\ x_2 &= a + 2\Delta x, \\ &\vdots \\ x_k &= a + k\Delta x, \\ &\vdots \\ x_n &= a + n\Delta x = b. \end{aligned}$$

The kth subinterval is $[x_{k-1}, x_k]$, and the width of each subinterval is Δx.

We choose x_k^* to be some x value in the kth subinterval, and then we choose the height of the kth rectangle to be $f(x_k^*)$.

We can choose x_k^* to be *any* number in the kth subinterval, but some common choices are:

- Choose $x_k^* = x_{k-1}$, the **left endpoint** of the kth subinterval
- Choose $x_k^* = x_k$, the **right endpoint** of the kth subinterval
- Choose $x_k^* = (x_{k-1} + x_k)/2$, the **midpoint** of the kth subinterval

The sum of the areas of all our rectangles is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

and this sum is called a **Riemann sum**.

EXAMPLE 1. Evaluate the left endpoint Riemann sum for the function $f(x) = x^2$ on the interval $[2, 5]$ using $n = 6$ subintervals.

Sigma notation

The Greek letter Σ (capital sigma) means 'sum of'.

For example, $\displaystyle\sum_{k=1}^{5} k^2$ means the sum of k^2 as k goes from 1 to 5.

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \qquad = \qquad 55.\end{aligned}$$

Another example:

$$\sum_{k=7}^{10} 2^k \qquad \text{means} \qquad 2^7 + 2^8 + 2^9 + 2^{10}.$$

In expressions like those above, the k is called the 'index'. It doesn't have to be k.

Notice that $\displaystyle\sum_{m=1}^{5} m^2$ means $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, which is 55 like before.

The index (whether it's k or m or something else) is sometimes called a 'dummy variable'.

Notice that when we evaluate $\displaystyle\sum_{k=1}^{5} k^2$ or $\displaystyle\sum_{m=1}^{5} m^2$, we 'use up' the k or m.

Properties of sums

$$\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k \qquad \text{if } c \text{ doesn't depend on } k$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

EXAMPLE 2. Evaluate the following sums.

(i) $\sum_{k=1}^{100} k$ (the answer should just be a number)

(ii) $\sum_{k=1}^{n} k$ (the answer should be a formula containing n)

The answer to **(ii)** is a formula for the sum of the first n positive integers.

It's possible to get similar formulas for things like $\sum_{k=1}^{n} k^2$ or $\sum_{k=1}^{n} k^3$.

A Riemann sum using n equal subintervals can be abbreviated using sigma notation.

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

The *official definition* of a definite integral is a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

if the limit exists independently of the choice of x_k^*.

Informally, we can say $f(x)$ is the 'signed height' and dx is a small change in x.

Fortunately, **there are shorter ways to evaluate integrals in practice** that don't involve calculating the limit of a Riemann sum.

EXAMPLE 3. Evaluate the definite integral using our knowledge of geometry.

$$\int_2^7 (x-4)\,dx$$

Properties of integrals

Previously, we assumed $a < b$. We now adopt the following conventions:

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

(Informally, integrating from a larger x value to a smaller x value is like going backwards, and integrating from a to a means integrating over an interval of width 0.)

We also have the following. (The first two resemble 'properties of sums' we saw earlier.)

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx \qquad \text{if } c \text{ is any constant}$$

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$

In that last rule, we do not require p to be between a and b!

In Week 1, we have discussed what integrals *are*, conceptually.

In Week 2 (and later), we will discuss how to **calculate integrals in practice**.

It turns out that we can calculate many integrals efficiently using **antiderivatives**.

Evaluating integrals using antiderivatives

In Calc I, you learn lots of derivative facts. You still need to know those facts in Calc II!

Every derivative fact can be rephrased as an antiderivative fact. For example, we know that if $F(x) = \sin x$, its derivative is $F'(x) = f(x) = \cos x$.

This means that the *antiderivative* of $f(x) = \cos x$ is $F(x) = \sin x + C$. (Functions have more than one antiderivative, but all antiderivatives of a given function differ by a constant.)

We write $\int f(x)\,dx$ for the antiderivative of $f(x)$. This is called an indefinite integral.

Here are some common antiderivatives that are consequences of derivative facts from Calc I.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \text{ is any constant EXCEPT } -1$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

USEFUL FACT: If $F(x)$ is any antiderivative of $f(x)$, then we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$