WHY can we evaluate integrals using antiderivatives?

Remember that the variable of integration is a 'dummy variable' that goes away after you perform the integration. For instance, $\int_2^5 t^2\,dt$ is just a number, not a function of t.

It's meaningful to define the **area function**

$$A(x) = \int_c^x f(t)\,dt$$

where c is a constant and we let x vary. This is a function of x, which can be loosely described as 'the net area under the graph of f up to x.'

It's then also meaningful to ask about the **derivative** of the area function. What is it?

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = \cdots = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \cdots$$

It turns out that $A'(x)$ is just $f(x)$! In other words, $A(x)$ is **an antiderivative** of $f(x)$.

Now, what if we happen to just 'know' a function $F(x)$ that's an explicit antiderivative of $f(x)$? Any two antiderivatives of the same function must differ by a constant, so we have

$$\int_c^x f(t)\,dt = F(x) + K \qquad \text{for some constant } K.$$

We then have

$$\int_c^b f(t)\,dt = F(b) + K$$

$$\int_c^a f(t)\,dt = F(a) + K$$

so $\int_a^b f(t)\,dt = \int_c^b f(t)\,dt - \int_c^a f(t)\,dt = \big(F(b)+K\big) - \big(F(a)+K\big) = F(b) - F(a)$.

Fundamental Theorem of Calculus, Part 1

If $A(x) = \displaystyle\int_c^x f(t)\,dt$, then $A'(x) = f(x)$.

Fundamental Theorem of Calculus, Part 2

If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We sometimes abbreviate $F(b) - F(a)$ as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

That notation obeys the following properties:

$$\Big[F(x) + G(x)\Big])a^b = \Big[F(x)\Big]_a^b + \Big[G(x)\Big]_a^b$$
$$\Big[cF(x)\Big]_a^b = c\Big[F(x)\Big]_a^b$$
$$\Big[-F(x)\Big]_a^b = \Big[F(x)\Big]_b^a$$

There can be more than one correct way to do arithmetic, but these shortcuts can help you!

EXAMPLE 1: Evaluate the definite integral.

$$\int_{-1}^{2} (x^2 - x - 2)\,dx$$

More shortcuts: even and odd functions

In precalculus, you learn about 'even' and 'odd' functions.

If $f(-x) = f(x)$ for all x, then f is called 'even'.

If $f(-x) = -f(x)$ for all x, then f is called 'odd'.

What do these conditions say about the graph of f?

EXAMPLE 2: Consider the following definite integral.

$$\int_{-2}^{2} (x^3 + x^2)\, dx$$

(i) Using the Fundamental Theorem of Calculus, what's the answer?

(ii) Are there any shortcuts?

Average value of a function

What's the average value of something that changes *continuously*?

The average value of the function $f(x)$ on the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

(Integrate, then divide by the width of the interval)

Fact (Mean Value Theorem for integrals)

If f is a continuous function on an interval $[a, b]$, then f must attain its average value at least once. That is, we must have

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

for some c in the interval (a, b).

(The Mean Value Theorem is important when proving things.)

Substitution

Many times, we *don't* know an antiderivative by just 'looking'. For example,

$$\int 3x^2(x^3+11)^5\,dx = ?$$

The method of **substitution** is like the chain rule in reverse.

FACT: $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot g'(x).$

Therefore $\displaystyle\int f'\big(g(x)\big)\cdot g'(x)\,dx = f\big(g(x)\big)+C.$

This is 100% true, *but* messy to use if written this way.

There is **other notation** we use which is easier to work with.

Namely, use u as shorthand for $g(x)$, and use du as shorthand for $g'(x)\,dx$.

ROUGH GUIDELINE: Choose u to be an 'inside' function, and hope du also appears.

EXAMPLE 3: Evaluate the indefinite integral.

$$\int \sin^4 x \cos x \, dx$$

Substitution in a **definite** integral

General fact:

$$\int_{x=a}^{x=b} f'\big(g(x)\big) \cdot g'(x)\, dx = f\big(g(b)\big) - f\big(g(a)\big) = \Big[f(u)\Big]_{u=g(a)}^{u=g(b)}$$

This is true, *but* we want a *procedure* that we can follow when evaluating definite integrals by substitution.

EXAMPLE 4: Evaluate the definite integral.

$$\int_1^4 \frac{1}{(3x+1)^2}\, dx$$

Note: When evaluating a **definite** integral, the final answer is just a **number**. So, if we do substitution, we don't need to go back to the old variable!

Defining the natural log as an integral

How do we define natural logs and the number e?

Either we define e first, or we define natural logs first.

If we define e first, we have to decide on a precise definition for e, and we also have to define e^x for irrational numbers.

An attractive alternative is to define natural logs first, using integrals.

We officially **define** $\ln x$ to be $\displaystyle\int_1^x \frac{1}{t}\,dt$.

Then the fact that $\displaystyle\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$ follows from (part 1 of) the Fundamental Theorem of Calculus.

Then $\ln(xy) = \displaystyle\int_1^{xy} \frac{1}{t}\,dt$. It's possible to *prove* that this is equal to $\ln x + \ln y$ with the help of substitution.

For us, definitions are less important than the **properties** of logarithms.

$$\ln(xy) = \ln x + \ln y$$
$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$
$$\ln(x^r) = r\ln x$$
$$\frac{d}{dx}\big(\ln|x|\big) = \frac{1}{x}$$

We can then officially define exponential functions with other bases.

If b is any positive number, then

$$b^x = (e^{\ln b})^x = e^{x \ln b}.$$

As a consequence, we have

$$\frac{d}{dx}\big(b^x\big) = b^x \ln b.$$

Here's an integration problem where we can use substitution and logarithms.

EXAMPLE 5: Evaluate the integral.

$$\int_0^2 \frac{x}{x^2+1}\, dx$$