Velocity and net change

It's common to write $s(t)$ for **position** as a function of time. If you are allowed to move forward or backward along a line, then position can be positive or negative.

Velocity is the derivative of position with respect to time: $v(t) = s'(t)$. Velocity can be positive or negative.

Displacement means (net) change in position. Your displacement between time $t = a$ and time $t = b$ is given by $s(b) - s(a)$.

By the Fundamental Theorem of Calculus, we have

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

(displacement is the integral of velocity).

Speed is the absolute value of velocity. In other words, speed ignores the direction of travel.

The **total distance traveled** (ignoring direction) is the integral of speed:

$$\int_a^b |v(t)|\,dt$$

(this can be justified by adding up all the little distances you travel). To evaluate an integral like that, you need to find the intervals where $v(t)$ is positive or negative.

The position function is an antiderivative of the velocity function. If you have an 'initial value', you can determine *which* antiderivative.

Acceleration is the derivative of velocity with respect to time (so it's also the second derivative of the position function). Acceleration can be positive or negative.

The velocity function is an antiderivative of the acceleration function. If you have an 'initial value', you can determine *which* antiderivative.

EXAMPLE 1: Find the position function and velocity function of an object moving along a straight line with the following acceleration, initial velocity, and initial position.

$$a(t) = -32, \qquad v(0) = 50, \qquad s(0) = 12$$

Regions between curves

Suppose f and g are two continuous functions. If $f(x) \le g(x)$ on the interval $[a, b]$, then the area bounded by the graphs of f and g on that interval is

$$\text{Area} = \int_a^b \big(\text{top} - \text{bottom}\big)\, dx = \int_a^b \big(g(x) - f(x)\big)\, dx.$$

This can be justified by slicing the region into a large number of skinny rectangles whose height is $g(x) - f(x)$ and whose width is dx.

EXAMPLE 2: Find the area bounded by the graphs of $y = x$ and $y = x^2 - 2$.

Integrating with respect to y

It's perfectly possible to use y as the input variable and have x be a function of y.

If $f(y)$ and $g(y)$ are two functions satisfying $f(y) \leq g(y)$ on the interval $c \leq y \leq d$, then the graph of $x = f(y)$ will be the 'left curve' and the graph of $x = g(y)$ will be the 'right curve'. (The larger outputs are the larger x values, which are located further right.)

When we integrate functions of y, we call that 'integrating with respect to y'. In **any** integral, it is always the **input** variable that increases by a small amount. So if we're integrating with respect to y, then y changes by a small amount dy.

Geometrically, this means cutting our region into slices that are 'skinny in the y direction'.

In situations like this, we have

$$\text{Area} = \int_a^b \big(\text{right} - \text{left}\big)\, dy = \int_a^b \big(g(y) - f(y)\big)\, dy.$$

EXAMPLE 3: Find the area bounded by the graphs of $x = 2y^2$ and $y = 2 - x/2$.

Volume by slicing

Suppose a solid object extends in the x direction from $x = a$ to $x = b$. We can express the total volume of the solid as the sum of the volumes of a large number of thin 'slices', each of thickness dx. Moreover, the volume of each slice is its area multiplied by its thickness.

$$\text{Volume} = \int_a^b (\text{volume of typical slice}) = \int_a^b (\text{area of typical slice})\, dx.$$

One type of solid we study a lot is a 'solid of revolution'. This is a solid formed when a region in a two-dimensional plane is revolved (out of the plane) around a line in the plane, creating a three-dimensional object.

For example, suppose f is a continuous nonnegative function on the interval $[a, b]$, and let R be the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$. If the region R is revolved around the x-axis, this forms a solid of revolution whose cross-sections are **disks**.

In this case, the radius of a typical cross-section is $f(x)$, so the area of a typical cross-section is $\pi\big(f(x)\big)^2$.

$$\text{Volume by disk method about the } x\text{-axis} = \int_a^b \pi\big(f(x)\big)^2\, dx$$

EXAMPLE 4: Let R be the region bounded by the curves $y = 2 - 2x$, $y = 0$, and $x = 0$. Find the volume obtained when R is revolved around the x-axis.

Washer method

Suppose f and g are *two* continuous nonnegative functions on the interval $[a, b]$, and suppose $f(x) \leq g(x)$ on $[a, b]$ (so f is the 'bottom' function and g is the 'top' function). Let A be the region bounded by the graphs of f and g on the interval $[a, b]$, and suppose A is revolved around the x-axis.

This time, we can slice the region A into skinny rectangles whose height is $g(x) - f(x)$ and whose width is dx. What happens when one of these 'typical rectangles' is revolved around the x-axis? What kind of three-dimensional shape is formed?

In this case, the typical slice of the solid of revolution is a **washer**; i.e., a disk with a disk-shaped hole in it. What's the area of a washer? It's the area of the big disk minus the area of the little disk. So it will have the form $\pi R^2 - \pi r^2$. In the scenario described on this page, R will be $g(x)$ (the top function) and r will be $f(x)$ (the bottom function).

$$\text{Volume by washers around the } x\text{-axis} = \int_a^b (\pi R^2 - \pi r^2)\, dx$$

where $R =$ top function and $r =$ bottom function.

EXAMPLE 5: Let A be the region bounded by the curves $y = x$ and $y = x^{1/4}$. Find the volume obtained when A is revolved around the x-axis.

Washer method when revolving around the y-axis

The washers considered on the previous page were formed by revolving a vertical rectangle with width dx around a horizontal line such as the x-axis. In other words, the skinny rectangles being revolved were perpendicular to the line we were revolving around (the axis of revolution).

As mentioned previously, it's also possible to use y as the input variable. If we do that, we might have two functions of y, say $x = f(y)$ and $x = g(y)$, one of which is the left curve and one of which is the right curve. When we integrate, it's always the input variable that increases by a small amount, so in this case we would integrate with respect to y and our skinny rectangles will be horizontal rectangles with height dy.

If those skinny horizontal rectangles are revolved around a *vertical* line, then again the skinny rectangles are perpendicular to the axis of revolution, so again the typical slices are washers.

EXAMPLE 6: Let A be the region bounded by the curves $x = \sec y$, $x = 2$, $y = 0$, and $y = \pi/3$. Find the volume obtained when A is revolved around the y-axis.

Revolving around other lines

Sometimes a solid of revolution is formed by revolving around a line other than one of the coordinate axes.

If the slices of our region (skinny rectangles) are perpendicular to the axis of revolution, we can use the washer method. Either revolving vertical rectangles around some horizontal line, or revolving horizontal rectangles around some vertical line.

We can still think of the area of a washer as a larger disk minus a smaller disk. So we can still use the formula

$$\text{Volume by washers} = \int_a^b (\pi R^2 - \pi r^2)\,dx.$$

This time, we have

$$R = \text{distance from 'far' curve to axis of revolution,}$$
$$r = \text{distance from 'near' curve to axis of revolution.}$$

EXAMPLE 7: Let A be the region bounded by the curves $y = 2\sin x$ and $y = 0$ on the interval $[0, \pi]$. Find the volume obtained when A is revolved around the line $y = -2$.