Volume by cylindrical shells

When we computed volumes of revolution using *washers*, we had rectangular slices of area that were *perpendicular* to the axis we were revolving around.

By contrast, if our rectangular slices of area are **parallel** to the axis we're revolving around, then revolving a rectangular slice generates a different shape, called a **(cylindrical) shell**.

For example, suppose the region bounded by two curves of the form $y = f(x)$ and $y = g(x)$ is revolved around the y-axis. Since x is the input variable, we have tall skinny rectangular slices with thickness dx. When one of those rectangular slices is revolved around the y-axis, we get a cylindrical shell.

All volumes of revolution can be computed in the following way:

$$\text{Total volume} = \int (\text{volume of typical piece}) = \int (\text{area of typical piece}) \cdot (\text{thickness})$$

The formula for the area of a cylindrical shell is different from that for of a washer.

Fact: The formula for the side surface area of a cylinder is $2\pi rh$, where r is the cylinder's radius and h is the cylinder's height.

General formula for volume by cylindrical shells:

$$\text{Volume by shells} = \int 2\pi rh \cdot \text{thickness}$$

If the area between $y = f(x)$ and $y = g(x)$ (say $f \leq g$) is revolved around the y-axis, then

$$\text{Volume by shells around } y\text{-axis} = \int 2\pi x\big(g(x) - f(x)\big)\, dx$$

EXAMPLE 1: Let A be the region bounded by $y = 3x - x^2$ and the x-axis. Find the volume obtained when the region A is revolved around the y-axis.

EXAMPLE 2: Let A be the region bounded by $x = 2y - y^2$ and $x = y$. Find the volume obtained when the region A is revolved around the x-axis.

When to use washers and when to use shells?

In a volume of revolution problem, you start with a two-dimensional region in the plane.

If that region's boundaries are of the form $y = f(x)$, then x is the input variable which changes by a small amount dx, and our 'area slices' are rectangles that are skinny in the x direction.

If that region's boundaries are of the form $x = f(y)$, then y is the input variable which changes by a small amount dy, and our 'area slices' are rectangles that are skinny in the y direction.

Next, the two-dimensional region is revolved around a line (which could be horizontal or vertical), called the 'axis of revolution'.

If our 'area slices' are **PERPENDICULAR** to the axis of revolution, we use **WASHERS**.

If our 'area slices' are **PARALLEL** to the axis of revolution, we use **SHELLS**.

Notice that there are actually *four* possibilities for a volume of revolution problem.

- $y = f(x)$, revolve around a horizontal line
- $y = f(x)$, revolve around a vertical line
- $x = f(y)$, revolve around a horizontal line
- $x = f(y)$, revolve around a vertical line

Summary of volume by washers and volume by shells

Volume by washers is

$$\begin{aligned}\int(\text{volume of washer}) &= \int(\text{area of washer})\cdot(\text{thickness})\\ &= \int(\pi R^2-\pi r^2)\cdot(\text{thickness})\end{aligned}$$

where R is the big radius and r is the small radius. More specifically,

$$\begin{aligned}R &= \text{distance from `far curve' to axis of revolution}\\ r &= \text{distance from `near curve' to axis of revolution}\end{aligned}$$

Volume by shells is

$$\begin{aligned}\int(\text{volume of shell}) &= \int(\text{area of shell})\cdot(\text{thickness})\\ &= \int 2\pi rh\cdot(\text{thickness})\end{aligned}$$

where r is the radius of the shell and h is the height of the shell. More specifically,

$$\begin{aligned}r &= \text{distance from `typical slice' to axis of revolution}\\ h &= \text{distance from `big curve' to `small curve'}\end{aligned}$$

EXAMPLE 3: Let A be the region bounded by $y = x^2$, $x = 1$, and $y = 0$. Find the volume obtained when the region A is revolved around the line $x = -2$.

Length of curves

The length of a curve can be approximated by a large number of short diagonal line segments.

The length of one of those short segments (the 'element' of arc length) is

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

The total length of a curve can be written as

$$\begin{aligned}\text{Total length} &= \int (\text{length of typical piece}) \\ &= \int ds \\ &= \int \sqrt{(dx)^2 + (dy)^2} \\ &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\end{aligned}$$

If you want more explanation of that last step:

$$\begin{aligned}\sqrt{(dx)^2 + (dy)^2} &= \sqrt{\left(\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}\right)(dx)^2} \\ &= \sqrt{\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}}\sqrt{(dx)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\end{aligned}$$

You don't have to **just** memorize $\sqrt{1 + (dy/dx)^2}$. There's a reason.

If we need to compute arc length when x is a function of y, the formula is

$$\text{Total length} = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

This is just another rearrangement of $\sqrt{(dx)^2 + (dy)^2}$. We interchanged x and y.

EXAMPLE 4: Find the length of the portion of the curve

$$y = \frac{x^2}{2} - \frac{\ln x}{4}$$

between $x = 1$ and $x = 3$.

Surface area

When a curve of the form $y = f(x)$ is revolved around the x-axis, it generates a surface. The typical piece of surface area is a 'ribbon' (called the 'frustum' of a cone) whose area is $2\pi r\, ds$, where r is the radius of the ribbon, and ds is the same 'element of arc length' we saw before.

This means that in the situation described above, we have

$$\begin{aligned} \text{Total surface area} &= \int (\text{surface area of typical piece}) \\ &= \int 2\pi r\, ds \\ &= \int 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\, dx \end{aligned}$$

EXAMPLE 5: The portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ is revolved around the x-axis. Find the area of the surface generated.

Mass of a one-dimensional object

Suppose we have a one-dimensional object, such as a wire or rod, that has variable density, i.e., the density can be different at different points on the wire. For instance, x might stand for the position along the wire in centimeters, and ρ might stand for the density in grams per centimeter. If the density can depend on the position or location along the wire, then ρ is a function of x.

The total mass of the wire is given by

$$\begin{aligned}\text{Total mass} &= \int (\text{mass of typical small piece}) \\ &= \int (\text{density of small piece}) \cdot (\text{width of small piece}) \\ &= \int \rho(x)\, dx\end{aligned}$$

EXAMPLE 6: Find the total mass of a thin bar whose density at position x is given by $\rho(x) = x\sqrt{1 - x^2}$ for $0 \leq x \leq 1$.