There are some integrals where we're expected to just 'know' the antiderivative (and there are no 'steps'). For example:

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int \cos x\,dx = \sin x + C$$

Now, we are going to study integrals that *do* require several steps. For example:

$$\int x\cos x\,dx = ??$$

(If $f(x) = x\cos x$ is the 'child' function, what's the correct 'parent'?)

We could try to 'guess' an answer, but *our guess might be wrong!*

Integration by parts is the closest thing we have to the reverse of the product rule.

General fact: $\dfrac{d}{dx}(uv) = u'v + uv'$

Therefore $\displaystyle\int (u'v + uv')\,dx = uv + C$

Equivalently, $\displaystyle\int u'v\,dx \;+\; \int uv'\,dx = uv + C$

Rearranging gives us $\displaystyle\int uv'\,dx = uv - \int u'v\,dx$, or more briefly:

$$\int u\,dv = uv - \int v\,du$$

This is the **integration by parts** formula. It transforms an integral into a new integral.

EXAMPLE 1: Evaluate the integral.

$$\int x \cos x \, dx$$

Note: To use integration by parts, the original integral needs to be *equal* to $\int u \, dv$.

EXAMPLE 2: Evaluate the integral.

$$\int xe^x \, dx$$

Once again, the formula for integration by parts is: $\int u\,dv = uv - \int v\,du$

Integration by parts with a **definite** integral looks like

$$\int_{x=a}^{x=b} u\,dv = \Big[uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

EXAMPLE 3: Evaluate the integral.

$$\int_1^2 x \ln x\,dx$$

Sometimes you need to integrate by parts more than once.

EXAMPLE 4: Evaluate the integral.

$$\int x^2 \cos x \, dx$$

Integration by parts with ‘deja vu’

EXAMPLE 5: Evaluate the integral.

$$\int e^x \sin x \, dx$$

Trigonometric integrals

This refers to integrals of powers of trig functions.

Some are easy, some are slightly tricky, some are very tricky.

Remember: Any time you know a derivative fact, you also know an antiderivative fact.

Derivatives of the six trig functions:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

This means there are *some* trig integrals you should just 'know'.

$$\int \cos x\, dx = \sin x + C$$
$$\int \sin x\, dx = -\cos x + C$$
$$\int \sec^2 x\, dx = \tan x + C$$
$$\int \sec x \tan x\, dx = \sec x + C$$

What about other integrals involving trig functions?

EXAMPLE 6: Evaluate the integral.

$$\int \cos 2x\, dx$$

EXAMPLE 7: Evaluate the integral.

$$\int \cos^3 x \sin x \, dx$$

EXAMPLE 8: Evaluate the integral.

$$\int \sec^4 x \, dx$$

Strategies for integrating powers of trig functions:

- Look for a 'candidate' for du (if doing substitution)
- Remember, **even** powers are easy to rewrite!

EXAMPLE 9: Evaluate the integral.

$$\int \cos^3 x \, dx$$

Let's discuss strategies for the following integrals.

$$\int \sin^5 x \, dx = ??$$

$$\int \sin^3 x \cos^3 x \, dx = ??$$

$$\int \sec^2 x \tan x \, dx = ??$$

$$\int \sec^3 x \tan x \, dx = ??$$

$$\int \sec^4 x \tan^2 x \, dx = ??$$

Two more trig identities, related to the 'double angle' identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Do you *have* to remember these? Maybe not. But they are the most efficient way to integrate $\sin^2 x$ or $\cos^2 x$.

EXAMPLE 10: Evaluate the integral.

$$\int \sin^2 x \, dx$$

The fastest way is to start with a trig identity.

Another correct way is to use integration by parts with 'deja vu'.

EXAMPLE 11: Evaluate the integral.

$$\int \sec x \, dx$$

The fastest way is to remember a weird 'trick'.

Another correct way is to rewrite $\sec x$ as $1/\cos x$ and do some algebra.