Trigonometric substitution

Trig substitution is typically used for integrals containing

$$\sqrt{a^2 \pm x^2} \qquad \text{or} \qquad \sqrt{x^2 \pm a^2}$$

where a is a constant. For example:

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = ??$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = ??$$

$$\int \frac{1}{\sqrt{25x^2-4}}\,dx = ??$$

Guidelines for deciding on a trig substitution:

- If you see $\sqrt{a^2 - x^2}$, try $x = a\sin\theta$. (Why? Because you get $1 - \sin^2$, which is $\cos^2$.)
- If you see $\sqrt{a^2 + x^2}$, try $x = a\tan\theta$. (Why? Because you get $1 + \tan^2$, which is $\sec^2$.)
- If you see $\sqrt{x^2 - a^2}$, try $x = a\sec\theta$. (Why? Because you get $\sec^2 - 1$, which is $\tan^2$.)

Note that *you don't need to memorize these 'blindly'*. Instead:

- remember the Pythagorean identities
- remember that you want a *perfect square under the square root.*

EXAMPLE 1: Evaluate the integral.

$$\int \frac{dx}{\sqrt{9-x^2}}$$

This means the same thing as $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.

EXAMPLE 2: Evaluate the integral.

$$\int \frac{dx}{\sqrt{4x^2-49}}$$

Strategy: Replace $4x^2$ with 49(trig function)^2.

But *which* trig function? (Remember the Pythagorean identities!)

Trig substitution with a **definite** integral

EXAMPLE 3: Evaluate the integral.

$$\int_0^2 \frac{x^3}{\sqrt{x^2+4}}\,dx$$

Partial fractions

Partial fractions is a technique of *algebra*. It's something you can do to rational functions *before* you integrate.

You know how to add fractions by converting to a common denominator:

$$\frac{2}{5} + \frac{3}{7} \;=\; \frac{2}{5}\cdot\frac{7}{7} + \frac{3}{7}\cdot\frac{5}{5} \;=\; \frac{14}{35} + \frac{15}{35} \;=\; \frac{29}{35}$$

You can do the same thing if the denominators contain x.

$$\begin{aligned}\frac{2}{x-4} + \frac{3}{x+1} &= \frac{2(x+1)}{(x-4)(x+1)} + \frac{3(x-4)}{(x-4)(x+1)} \\ &= \frac{2x+2}{(x-4)(x+1)} + \frac{3x-12}{(x-4)(x+1)} \\ &= \frac{5x-10}{(x-4)(x+1)}\end{aligned}$$

Partial fractions is the *reverse* process of this.

EXAMPLE 4: Evaluate the integral.

$$\int \frac{3x-2}{(x-3)(x+4)}\,dx$$

Strategy: Try to 'split' $\dfrac{3x-2}{(x-3)(x+4)}$ into $\dfrac{\text{something}}{x-3} + \dfrac{\text{something}}{x+4}$.

EXAMPLE 4: Evaluate the integral.

$$\int \frac{3x-2}{(x-3)(x+4)}\,dx$$

KEY STEP: Write down

$$\frac{3x-2}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$$

and then do algebra to find A and B.

EXAMPLE 5: Evaluate the integral.

$$\int \frac{x^2 + 4x - 9}{x^3 - 6x^2 + 9x}\, dx$$

First step: How does the denominator factor?

EXAMPLE 6: Evaluate the integral.

$$\int \frac{5x^2 + x + 3}{x^3 + x}\,dx$$

First step: How does the denominator factor?

GUIDELINE:

If you have a repeated factor, you need different powers of that factor.

For example, if doing partial fractions on $\frac{6x+7}{(x+2)^2}$, then write:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

GUIDELINE:

If you have an irreducible quadratic factor, its numerator is linear.

For example, if doing partial fractions on $\frac{5}{(x+1)(x^2+1)}$, then write:

$$\frac{5}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Some integral facts:

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int \frac{1}{x+2}\, dx = \ln|x+2| + C$$

$$\int \frac{1}{x-3}\, dx = \ln|x-3| + C$$

CAREFUL, though. Note that $\int \frac{1}{5x-7}\, dx = \frac{1}{5}\ln|5x-7| + C$.

FACT:

$$\int \frac{1}{5x-7}\,dx = \frac{1}{5}\ln|5x-7| + C$$

Why the $\frac{1}{5}$ in front? Because when you take the derivative of $\ln|5x-7|$, you MUST use the chain rule!

FACT: $\displaystyle\int \frac{1}{x^2+1}\,dx = \arctan x + C$

Notice that this has NOTHING to do with logarithms!

We just 'know' this because we know $\frac{d}{dx}\arctan x = \frac{1}{x^2+1}$.

GENERAL RULE:

$$\int \frac{1}{u}\,du = \ln|u| + C$$

The du is a **very important part** of this rule!

The integral $\displaystyle\int \frac{1}{x^2+1}\,dx$ is NOT of the form $\displaystyle\int \frac{1}{u}\,du$!

However, the integral $\displaystyle\int \frac{2x}{x^2+1}\,dx$ is. (What's $\frac{d}{dx}\ln(x^2+1)$?)

EXAMPLE 7: Evaluate the integral.

$$\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$$

First step: Perform division to get a numerator with smaller degree.