Numerical integration

How can we evaluate a definite integral, such as the following?

$$\int_2^5 x^2\,dx$$

Since we *happen* to know an antiderivative of $f(x) = x^2$, we can do:

$$\left[\frac{x^3}{3}\right]_2^5 = \frac{1}{3}(5^3 - 2^3) = \frac{1}{3}(125 - 8) = \frac{117}{3} = 39$$

But there are many times when antiderivatives are difficult or impossible.

When first studying integrals, we learn that we can informally regard

$$\int_2^5 x^2\,dx$$

as the sum of $x^2 \cdot dx$ as x takes on 'every' value from 2 to 5, and dx is a tiny increase in x.

We can approximate an integral by dividing our interval $[a, b]$ into n equal subintervals, each of width $\Delta x = (b - a)/n$. For example, if we divide the interval $[2, 5]$ into $n = 300$ subintervals, then we have $\Delta x = (5 - 2)/300 = 0.01$. This gives us a total of 301 'grid points'

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_{299},\ x_{300}.$$

If we use the left endpoint of each subinterval then we use the 300 points $x_0, x_1, \ldots, x_{299}$.

If we use the right endpoint of each subinterval then we use the 300 points $x_1, x_2, \ldots, x_{300}$.

Using $n = 300$, the left-endpoint approximation of $\int_2^5 x^2\,dx$ is

$$2^2 \cdot 0.01 + (2.01)^2 \cdot 0.01 + (2.02)^2 \cdot 0.01 + \cdots + (4.98)^2 \cdot 0.01 + (4.99)^2 \cdot 0.01$$

and the right-endpoint approximation is

$$(2.01)^2 \cdot 0.01 + (2.02)^2 \cdot 0.01 + (2.03)^2 \cdot 0.01 + \cdots + (4.99)^2 \cdot 0.01 + 5^2 \cdot 0.01.$$

For a general function f, the left-endpoint approximation is

$$\Delta x \cdot \Big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-2}) + f(x_{n-1}) \Big)$$

and the right-endpoint approximation is

$$\Delta x \cdot \Big(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n) \Big)$$

The average of those two approximations is

$$\frac{\Delta x}{2} \cdot \Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \Big)$$

This is the **trapezoid rule**.

Things to notice:

- the 2 in the denominator beneath the Δx
- the 1,2,2,...,2,1 pattern of coefficients in front of the f's

Another formula for numerical integration is **Simpson's rule**.

$$\frac{\Delta x}{3} \cdot \Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \Big)$$

Things to notice:

- the 3 in the denominator beneath the Δx
- the 1,4,2,4,2,. . . ,2,4,1 pattern of coefficients in front of the f's

Here, n must be even. The pattern is $1, 4, 1$ or $1, 4, 2, 4, 1$ or $1, 4, 2, 4, 2, 4, 1$, etc.

For example: Trapezoid rule with $n = 6$ subintervals

$$\frac{\Delta x}{2} \cdot \Big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6) \Big)$$

and Simpson's rule with $n = 6$ subintervals

$$\frac{\Delta x}{3} \cdot \Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6) \Big)$$

These are approximations of $\int_a^b f(x)\,dx$. We could also increase n.

EXAMPLE 1: Use the Trapezoid rule and Simpson's rule to approximate

$$\int_0^{\pi} \sin^4 x \, dx$$

using $n = 6$ subintervals.

Improper integrals

Integrals that involve infinity are called 'improper'.

Type I: One or both of the *endpoints* is infinite.

Type II: The *function we're integrating* becomes infinite.

For either type of integral, we must 'work around' the infinity using limits.

EXAMPLE 2: Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$.

Strategy: We must instead evaluate $\displaystyle\int_0^M \frac{1}{x^2+1}\,dx$ and *then* let $M \to \infty$.

EXAMPLE 3: Evaluate the integral.

$$\int_1^\infty \frac{1}{x^{1.001}}\,dx$$

EXAMPLE 4: Evaluate the integral.

$$\int_1^\infty \frac{1}{x^{0.999}}\,dx$$

EXAMPLE 5: Evaluate the integral.

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$$

Note: Even though you don't see an infinity *symbol*, nevertheless this integral is improper because the *function* becomes infinite at $x = 4$. (Division by zero)

We must 'work around' $x = 4$. Integrate from 0 to M, *then* let $M \to 4$.

EXAMPLE 6: Evaluate the integral.

$$\int_0^4 \frac{1}{(4-x)^{3/2}}\,dx$$

When evaluating an improper integral, the last step is evaluating a limit.

Sometimes the limit exists, and sometimes it does not exist.

So some improper integrals have a value, and some do not.

If an improper integral evaluates to a number, then we say the improper integral **converges**. If it does not, then we say the improper integral **diverges**.