Sequences and series

A **sequence** is like an infinite **list**.

If the numbers in a sequence get as close as we like to a certain number, then we say the sequence **converges**. Otherwise, it **diverges**.

An example of a sequence:

$$\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\right\}$$

The numbers in this sequence get closer and closer to 0. (They get as close as we like.)

We say that this **sequence** converges to 0.

This sequence can be denoted

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

This means: The sequence whose nth term is $\frac{1}{n}$, as n goes from 1 to ∞.

Another example of a sequence:

$$\left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots\right\}$$

We can denote this sequence by $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$ or by $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$

This **sequence** converges to 1.

In general, we say the sequence $\{a_n\}$ converges to L if $\lim_{n\to\infty} a_n = L$.

For example, we have $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{n-1}{n} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

Two more examples:

The sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ does **not** converge. Neither does $\{(-1)^n\}_{n=0}^{\infty}$.

Some limits are easy, some are hard, and some are in between.

You sometimes have to use knowledge of limits from Calc I.

EXAMPLE: The sequence

$$\left\{\frac{\ln 2}{2},\ \frac{\ln 3}{3},\ \frac{\ln 4}{4},\ \frac{\ln 5}{5},\ \ldots\right\} = \left\{\frac{\ln n}{n}\right\}_{n=2}^{\infty}$$

converges (to 0), because $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Some less obvious examples: The sequence

$$\left\{n^{1/n}\right\}_{n=1}^{\infty} = \{1,\ 2^{1/2},\ 3^{1/3},\ 4^{1/4},\ \ldots\}$$

It turns out that $\lim_{n\to\infty} n^{1/n} = 1$. This sequence converges to 1.

Another example: The sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty} = \left\{2,\ \left(1+\frac{1}{2}\right)^2,\ \left(1+\frac{1}{3}\right)^3,\ \left(1+\frac{1}{4}\right)^4,\ \ldots\right\}$$

It turns out that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. This sequence converges to e.

FACT: A **bounded monotone** sequence must converge.

'Bounded': There exist numbers M_1 and M_2 such that all members of the sequence are between M_1 and M_2.

'Monotone': The sequence is **always increasing** or **always decreasing**.

The sequence $\{1, -1, 1, -1, 1, -1, \ldots\}$ is bounded but not monotone.

This sequence does not converge. (It does not have a single limit.)

The sequence $\{\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots\}$ is monotone but not bounded.

This sequence does not converge. (Its terms get arbitrarily large and don't approach a point on the number line.)

The sequence $\{1,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ \ldots\}$ is monotone (decreasing) and bounded, so it must converge. (In fact, it converges to 0.)

The sequence $\{0,\ 1/2,\ 2/3,\ 3/4,\ 4/5,\ \ldots\}$ is monotone (increasing) and bounded, so it must converge. (In fact, it converges to 1.)

The sequence $\{-1,\ 1/2,\ -1/3,\ 1/4,\ -1/5,\ \ldots\}$ is not monotone, but it is bounded, and it does happen to converge (to 0).

A **series** is very different from a **sequence**.

A sequence is an infinite list, whereas a **series** is an infinite **sum**.

SERIES ARE FAR MORE SUBTLE THAN SEQUENCES!!!

Just like when we started integrals, we use $\sum$ to mean 'sum of'.

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{means} \qquad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{means} \qquad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \text{means} \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

Note: 2^n is a lot bigger than n^2, so $\frac{1}{2^n}$ is a lot **smaller** than $\frac{1}{n^2}$.

Also, $\frac{1}{n^2}$ is smaller than $\frac{1}{n}$.

We could compare with the improper integrals $\int_1^{\infty} \frac{1}{x}\,dx$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$.

The **partial sums** of a series are the sums 'so far'.

S_n = the sum of the first n terms of the series.

So if the series is $\sum_{n=1}^{\infty} a_n$ then we have

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$

and so on.

The specific series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

is called the **harmonic series**. What are its partial sums?

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2} = 1.5$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = 1.5 + 0.333\ldots = 1.8333\ldots$$

TRICK for understanding the harmonic series:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$$

$$\frac{1}{3} + \frac{1}{4} \qquad \text{is a group of 2 terms}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \qquad \text{is a group of 4 terms}$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \qquad \text{is a group of 8 terms}$$

If we cleverly decide to group in that way, then each group adds up to **more** than $\frac{1}{2}$, and we **never run out of groups!**

There is **no upper bound** to the partial sums of the harmonic series.

THE HARMONIC SERIES DIVERGES!!!

THIS IS LITERALLY ONE OF THE MOST IMPORTANT FACTS IN THE ENTIRE COURSE!!!

The **sequence** $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to zero. The numbers in that **list** get arbitrarily small.

However, the **series** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ turns out to **diverge**. If we keep **adding** $\frac{1}{n}$, then the total gets arbitrarily large.

So it's possible to have a_n where the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0 but the series $\sum_{n=1}^{\infty} a_n$ diverges.

We could compare with the improper integrals $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx$ and $\displaystyle\int_1^{\infty}\frac{1}{x^2}\,dx$.

It's true that $\displaystyle\lim_{x\to\infty}\frac{1}{x} = 0$, and it's true that $\displaystyle\lim_{x\to\infty}\frac{1}{x^2} = 0$.

But $\frac{1}{x}$ and $\frac{1}{x^2}$ are **different functions** approaching 0 at different rates.

One of those improper integrals converges, and one of them diverges.
It's **not enough** to know that the function approaches 0.

For some special series, we can find a formula for the partial sums.

Geometric series: Each term is the previous term times a constant.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \frac{1}{324} + \cdots$$

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$$

General geometric series:

$$a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots = \sum_{n=1}^{\infty} a\cdot r^{n-1}$$

Partial sum = sum of the first n terms:

$$S_n = a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots + a\cdot r^{n-2} + a\cdot r^{n-1}$$

TRICK: What happens if we multiply this by r?

FACT: Suppose we have a geometric series

$$a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots$$

If $-1 < r < 1$, then the geometric series converges, and its sum is $\frac{a}{1-r}$.