If a **series** $\sum a_n$ has positive terms then its partial sums are increasing.

The big question is: Do those partial sums increase without bound, or do they have a limit? In other words, is the total infinite or finite?

It depends on the details of what a_n is!

THE INTEGRAL TEST

Suppose f is a positive continuous decreasing function on $[N, \infty)$.

Then the infinite series $\displaystyle\sum_{n=N}^{\infty} f(n)$ will converge if and only if the improper integral $\displaystyle\int_N^{\infty} f(x)\,dx$ converges.

EXAMPLE 1:

$f(x) = \dfrac{1}{x}$ is a positive continuous decreasing function on $[1, \infty)$.

So we can determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges by checking whether the integral $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ converges or diverges.

EXAMPLE 2:

$f(x) = \dfrac{1}{x^2}$ is a positive continuous decreasing function on $[1, \infty)$.

So we can determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges by checking whether the integral $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$ converges or diverges.

EXAMPLE 3:

$f(x) = \dfrac{1}{\sqrt{x}}$ is a positive continuous decreasing function on $[1, \infty)$.

So we can determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges by checking whether the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ converges or diverges.

The functions $f(x) = \dfrac{1}{x}$, $f(x) = \dfrac{1}{x^2}$, $f(x) = \dfrac{1}{\sqrt{x}}$ are all *different.*

$$\int_1^{\infty} \frac{1}{x}\, dx = \Big[\ln x\Big]_1^{\infty}$$

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \int_1^{\infty} x^{-2}\, dx = \left[\frac{x^{-1}}{-1}\right]_1^{\infty}$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \int_1^{\infty} x^{-1/2}\, dx = \left[\frac{x^{1/2}}{1/2}\right]_1^{\infty}$$

If p is constant, a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.

The three examples on the previous page are all p-series, as well as

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.02}} = 1 + \frac{1}{2^{1.02}} + \frac{1}{3^{1.02}} + \frac{1}{4^{1.02}} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.97}} = 1 + \frac{1}{2^{0.97}} + \frac{1}{3^{0.97}} + \frac{1}{4^{0.97}} + \cdots$$

If we apply the integral test to p-series, we get the following rule.

- If $p > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
- If $0 \leq p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

(Notice: If p gets bigger, then $\frac{1}{n^p}$ gets smaller.)

EXAMPLE 4: Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge or diverge?

Note: $x \ln x$ is a positive increasing function of x, so $\dfrac{1}{x \ln x}$ is a positive decreasing function of x, so we can use the integral test.

To determine whether the series converges or diverges, check whether the integral $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ converges or diverges.

Comparison tests for series

First, the **direct** comparison test.

Say $\sum a_n$ and $\sum b_n$ are series with positive terms, and say $a_n \leq b_n$.

If $\sum b_n$ converges, that 'forces' the smaller series $\sum a_n$ to converge.

If $\sum a_n$ diverges, that 'forces' the larger series $\sum b_n$ to diverge.

(Smaller than finite is finite, larger than infinite is infinite)

EXAMPLE 5: Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converge or diverge?

Notice that $\dfrac{1}{n^2+3}$ is positive. Also notice

$$n^2 + 3 > n^2$$
$$\frac{1}{n^2+3} < \frac{1}{n^2}$$
$$\sum \frac{1}{n^2+3} < \sum \frac{1}{n^2}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, because it's a p-series with $p = 2 > 1$.

By direct comparison, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges.

EXAMPLE 6: Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-3}$ converge or diverge?

Rough idea: If n is large, then n^2-3 is just 'slightly' less than n^2, so $\dfrac{1}{n^2-3}$ is 'close' to $\dfrac{1}{n^2}$.

If $a_n = \dfrac{1}{n^2-3}$ and $b_n = \dfrac{1}{n^2}$, how do we formalize the idea that a_n and b_n are 'close' to each other when n is large?

The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ is not 0 and not ∞,
then the two series $\sum a_n$ and $\sum b_n$ 'behave the same',
i.e., either they both converge or both diverge.

EXAMPLE 7: Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{3n^2+1}$ converge or diverge?