Alternating series and absolute convergence

Some series have a mixture of positive and negative terms.

For example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

If $a_n = \dfrac{(-1)^{n-1}}{n}$, then $\sum a_n$ is a series with positive and negative terms.

More specifically, the terms of that series **alternate** between positive and negative.

Alternating series test: Suppose we have an **alternating** series $\sum (-1)^n a_n$ where a_n is positive. If a_n decreases to zero, then the alternating series converges.

Question: Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge or diverge?

Question: Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converge or diverge?

An infinite series $\sum a_n$ is called **absolutely convergent** if the related (but different) series $\sum |a_n|$ is convergent.

For example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

To check whether the above series is absolutely convergent, we check whether the related series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is convergent or not.

Question: Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

absolutely convergent or not?

Another question: Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

absolutely convergent or not?

FACT: If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

In other words, absolute convergence implies convergence.

Informally, absolutely convergent is like ‘strongly convergent’ or ‘convergent in a way that’s hard to destroy’.

The Ratio Test

Suppose $\sum a_n$ is an infinite series, and define

$$R = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then:

- If $R < 1$, then the series $\sum a_n$ converges absolutely.
- If $R > 1$, then the series $\sum a_n$ diverges.

(Unfortunately, if $R = 1$, then you don't know what happens, and you have to try something else.)

Intuition behind the ratio test:

When you apply the ratio test, you're looking at the ratio of two consecutive terms of your series.

If that ratio approaches $\frac{1}{3}$ (for example) then the later terms of your series 'resemble' a geometric series with $r = \frac{1}{3}$.