WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #3

Due Tuesday January 30th, in Canvas

Question 1. Consider the vectors $\mathbf{u} = \langle 7, 0, 0\rangle$ and $\mathbf{v} = \langle 2, 3, 0\rangle$.

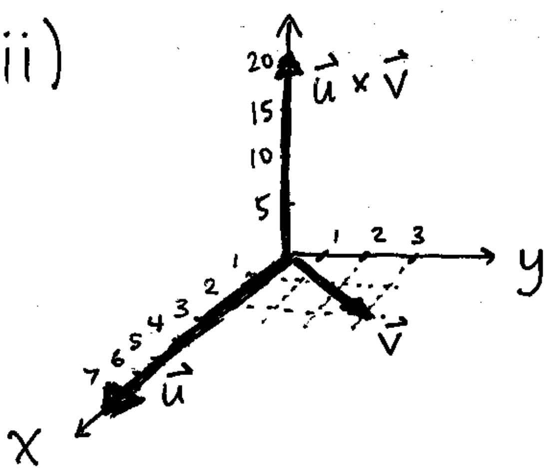
(i) Compute $\mathbf{u} \times \mathbf{v}$.

(ii) Draw a rough sketch of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ in $\mathbb{R}^3$.

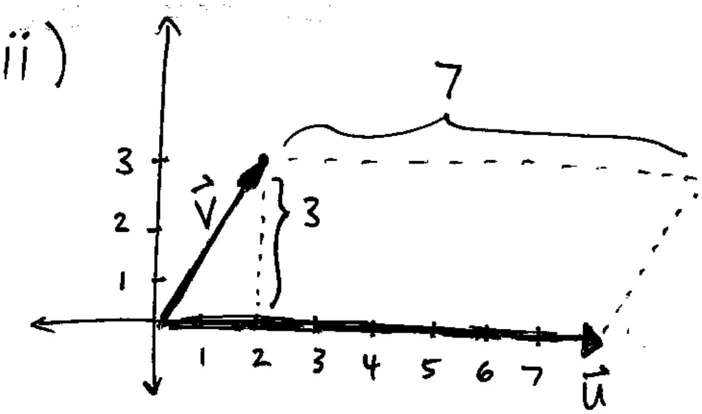
(iii) Draw a rough sketch of $\mathbf{u}$ and $\mathbf{v}$ in the xy-plane, and verify that $|\mathbf{u}\times\mathbf{v}|$ is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

(i) $$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \langle 0\cdot 0 - 0\cdot 3,\ 0\cdot 2 - 7\cdot 0,\ 7\cdot 3 - 0\cdot 2\rangle$$
$$= \langle 0-0,\ 0-0,\ 21-0\rangle$$
$$= \langle 0, 0, 21\rangle$$

(ii)



(iii)



$|\vec{u}\times\vec{v}| = |\langle 0,0,21\rangle| = 21$

Area of parallelogram = base$\cdot$height

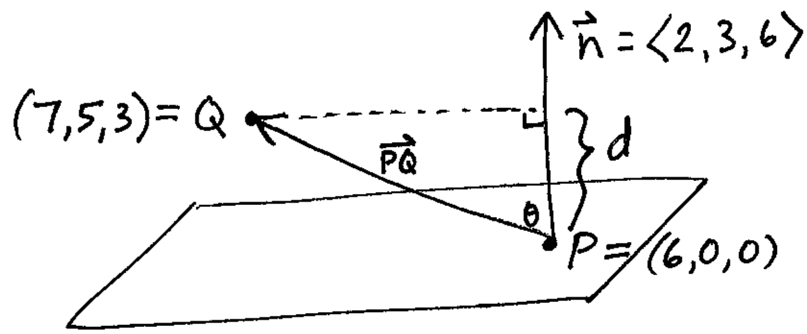
$= 7\cdot 3 = 21$ ✓

Question 2.

Find the distance from the point $\underbrace{(7, 5, 3)}_{Q}$ to the plane $2x + 3y + 6z = 12$.

$\vec{n} = \langle 2, 3, 6 \rangle$

An example of a point in the plane is $P = (6,0,0)$ since this satisfies $2x + 3y + 6z = 12$.



$\vec{PQ} = Q - P = \langle 1,5,3 \rangle$

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{d}{|\vec{PQ}|} \Rightarrow d = |\vec{PQ}|\cos\theta$$

We know $\vec{n} \cdot \vec{PQ} = |\vec{n}|\,|\vec{PQ}|\cos\theta$

$$\Rightarrow \frac{\vec{n} \cdot \vec{PQ}}{|\vec{n}|} = |\vec{PQ}|\cos\theta = d$$

$$\text{Distance} = \frac{\langle 2,3,6 \rangle \cdot \langle 1,5,3 \rangle}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2 + 15 + 18}{\sqrt{4 + 9 + 36}} = \frac{35}{\sqrt{49}}$$

$$= \frac{35}{7} = 5$$

Question 3. Do the lines

$$x = t, \qquad y = 2t + 1, \qquad z = 3t + 4$$

and

$$x = 2s - 2, \qquad y = 2s - 1, \qquad z = 3s + 1$$

intersect each other at only one point? If so, find a plane that contains both lines.

The lines intersect if there exist t and s satisfying the equations:

$$t = 2s - 2$$

$$2t+1 = 2s - 1 \Rightarrow 2(2s-2)+1 = 2s-1 \quad \Rightarrow 4s-3 = 2s-1$$

$$\Rightarrow 4s-2s = 3-1$$

$$3t+4 = 3s+1 \qquad 2s = 2 \quad \Rightarrow s = 1$$

$$\Rightarrow t = 2\cdot 1 - 2$$

$$= 2-2 = 0$$

We find that $t=0$, $s=1$ gives the point $(x,y,z) = (0,1,4)$ on both lines

A direction vector for the first line is $\vec{v} = \langle 1, 2, 3 \rangle$

A direction vector for the second line is $\vec{w} = \langle 2, 2, 3 \rangle$

A normal vector to the plane is $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} = \langle 0, 3, -2 \rangle$

So an equation of the plane will have the form $0x + 3y - 2z = d$

Now use the known point $(0,1,4)$ to find d.

$$0\cdot 0 + 3\cdot 1 - 2\cdot 4 = d$$

$$3 - 8 = d$$

$$d = -5$$

ANSWER: $3y - 2z = -5$

Question 4. Consider the curve in $\mathbb{R}^3$ defined by $\mathbf{r}(t) = \langle 10\cos t, 2\sin t, 1\rangle$.

(i) What kind of curve is it?

(ii) Find all points where the curve intersects the plane $y = 1$.

(i) An ellipse lying in the plane $z = 1$.

(Note that $x = 10\cos t$, $y = 2\sin t$ is similar to the usual parametrization of the unit circle except x and y have been scaled by different amounts)

(ii) $y = 1 \Rightarrow 2\sin t = 1 \Rightarrow \sin t = \frac{1}{2}$

$\Rightarrow t = \frac{\pi}{6}$ or $t = \frac{5\pi}{6}$ (could also add any multiple of 2π)

$$t = \frac{\pi}{6} \Rightarrow (x,y,z) = \left(10\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}, 1\right)$$
$$= \left(10\cdot\frac{\sqrt{3}}{2}, 2\cdot\frac{1}{2}, 1\right) = (5\sqrt{3}, 1, 1)$$

$$t = \frac{5\pi}{6} \Rightarrow (x,y,z) = \left(10\cos\frac{5\pi}{6}, 2\sin\frac{5\pi}{6}, 1\right)$$
$$= \left(10\cdot\frac{-\sqrt{3}}{2}, 2\cdot\frac{1}{2}, 1\right) = (-5\sqrt{3}, 1, 1)$$

(if we add any multiple of 2π to t we get those same points again)