WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #4

Due Tuesday February 6th, in Canvas

Question 1. The vector-valued function $\mathbf{r}(t) = \langle 8\cos 2t, 8\sin 2t\rangle$ defines a curve in $\mathbb{R}^2$.

(i) What shape is the curve? Why?

(ii) Calculate $\mathbf{r}'(t)$, and verify explicitly that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

(i) It's a circle (of radius 8).

Points on this curve satisfy $x^2+y^2 = 64\cos^2 2t + 64\sin^2 2t = 64$

The 2t means we travel "twice as fast" as the usual parametrization

(ii) $\vec{r}\,'(t) = \langle -8\sin 2t \cdot 2, 8\cos 2t \cdot 2\rangle$

$= \langle -16\sin 2t, 16\cos 2t\rangle$

$\vec{r}(t) \cdot \vec{r}\,'(t) = \langle 8\cos 2t, 8\sin 2t\rangle \cdot \langle -16\sin 2t, 16\cos 2t\rangle$

$= -128\cos 2t\sin 2t + 128\sin 2t\cos 2t$

$= 0$

Question 2. Find the length of the curve defined by $\mathbf{r}(t) = \langle t, t, t^{3/2} \rangle$ for $0 \leq t \leq 1$.

$$\vec{r}\,'(t) = \left\langle 1, 1, \tfrac{3}{2} t^{1/2} \right\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2 + \left(\tfrac{3}{2} t^{1/2}\right)^2} = \sqrt{1 + 1 + \tfrac{9}{4} t} = \sqrt{2 + \tfrac{9}{4} t}$$

$$\text{Length} = \int_{t=0}^{t=1} |\vec{r}\,'(t)|\,dt = \int_{t=0}^{t=1} \sqrt{2 + \frac{9t}{4}}\,dt = \int_{t=0}^{t=1} \sqrt{\frac{8+9t}{4}}\,dt$$

$$= \frac{1}{2} \int_{t=0}^{t=1} \sqrt{8+9t}\,dt.$$

Sub $u = 8+9t \Rightarrow du = 9dt \Rightarrow \frac{1}{9} du = dt$

If $t=0$ then $u = 8+0 = 8$

If $t=1$ then $u = 8+9 = 17$

$$\frac{1}{2} \int_{u=8}^{u=17} \sqrt{u} \cdot \frac{1}{9}\,du = \frac{1}{18} \int_{u=8}^{u=17} u^{1/2}\,du = \frac{1}{18} \left[\frac{u^{3/2}}{3/2} \right]_{u=8}^{u=17}$$

$$= \frac{1}{\cancel{18}_{\,9}} \cdot \frac{\cancel{2}}{3} \left[u^{3/2} \right]_{u=8}^{u=17} = \frac{1}{27} \left(17^{3/2} - 8^{3/2} \right)$$

Question 3 Consider the curve in $\mathbb{R}^2$ defined by $\mathbf{r}(t) = \langle 3t^2 - 1, 4t^2 + 5\rangle$.

(i). Explicitly find the arc length function $s(t) = \int_0^t |\mathbf{r}'(u)|\, du$.

(ii). Find the inverse of the function in **(i)**, i.e. write t as a function of s.

(iii). Rewrite the curve using s as the parameter. What type of curve is it?

(i) $\vec{r}(u) = \langle 3u^2 - 1, 4u^2 + 5\rangle \Rightarrow \vec{r}\,'(u) = \langle 6u, 8u\rangle$

$\Rightarrow |\vec{r}\,'(u)| = \sqrt{(6u)^2 + (8u)^2} = \sqrt{36u^2 + 64u^2} = \sqrt{100u^2} = 10u$ if $u \geq 0$

$s = s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t 10u\, du = \left[5u^2\right]_{u=0}^{u=t} = 5t^2$

$\boxed{s = s(t) = 5t^2}$

(ii) $s = 5t^2 \Rightarrow \frac{s}{5} = t^2 \Rightarrow t = \pm\sqrt{\frac{s}{5}}$ $\quad t = \sqrt{\frac{s}{5}}$ if $t \geq 0$

(iii) If $t^2 = \frac{s}{5}$ then $\vec{r} = \langle 3t^2 - 1, 4t^2 + 5\rangle$

$= \langle 3\frac{s}{5} - 1, 4\frac{s}{5} + 5\rangle$

$\vec{r} = \langle \frac{3}{5}s - 1, \frac{4}{5}s + 5\rangle$

or $\vec{r} = \langle -1, 5\rangle + s\langle \frac{3}{5}, \frac{4}{5}\rangle$

This is a <u>LINE</u> through $\langle -1, 5\rangle$ in the direction of $\langle \frac{3}{5}, \frac{4}{5}\rangle$

(or technically a half-line if $s \geq 0$)