WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #7

Due Tuesday March 5th, in Canvas

Question 1. Find an equation of the tangent plane to the surface

$$x^4 + y^5 + z^7 = 2$$

at the point $(-1, 0, 1)$.

$$f(x,y,z) = x^4 + y^5 + z^7 \quad \begin{aligned} &\nearrow f_x = 4x^3 \\ &\rightarrow f_y = 5y^4 \\ &\searrow f_z = 7z^6 \end{aligned}$$

$$\nabla f = (4x^3, 5y^4, 7z^6)$$

At the point $(-1,0,1)$ we have $\nabla f = (-4, 0, 7)$

The gradient is <u>normal</u> to the tangent plane.

Equation of tangent plane is

$$-4(x-(-1)) + 0(y-0) + 7(z-1) = 0$$

or $-4(x+1) + 7(z-1) = 0$

or $-4x - 4 + 7z - 7 = 0$

or $-4x + 7z - 11 = 0$ or $-4x + 7z = 11$

Question 2. Find an equation of the tangent plane to the surface

$$xy\sin z = 1$$

at the point $(1, 2, \pi/6)$.

$f(x,y,z) = xy \sin z$

$f_x = 1 \cdot y \cdot \sin z \quad f_y = x \cdot 1 \cdot \sin z \quad f_z = x \cdot y \cdot \cos z$

$\nabla f = (y \sin z, x \sin z, xy \cos z)$

$$\nabla f\left(1, 2, \tfrac{\pi}{6}\right) = \left(2 \underbrace{\sin\tfrac{\pi}{6}}_{1/2}, 1 \underbrace{\sin\tfrac{\pi}{6}}_{1/2}, 2 \underbrace{\cos\tfrac{\pi}{6}}_{\sqrt{3}/2}\right) = \left(1, \tfrac{1}{2}, \sqrt{3}\right)$$

Equation of tangent plane will have the form $1x + \frac{1}{2}y + \sqrt{3}z = d$

Use given point to find d.

$$\underbrace{1 \cdot 1}_{1} + \underbrace{\tfrac{1}{2} \cdot 2}_{1} + \sqrt{3} \cdot \tfrac{\pi}{6} = d \Rightarrow d = 2 + \frac{\pi\sqrt{3}}{6} \quad \text{or} \quad \frac{12 + \pi\sqrt{3}}{6}$$

Answer: $x + \frac{1}{2}y + \sqrt{3}z = \dfrac{12 + \pi\sqrt{3}}{6}$

or $2x + y + 2\sqrt{3}z = \dfrac{12 + \pi\sqrt{3}}{3}$

or $6x + 3y + 6\sqrt{3}z = 12 + \pi\sqrt{3}$

Question 3. Find the linear approximation to the function

$$f(x, y) = xy + x - y$$

at the point $(2, 3)$, and use it to estimate $f(2.03, 2.99)$.

$x_0 = 2 \qquad y_0 = 3$

$z_0 = f(x_0, y_0) = f(2,3) = \underbrace{2 \cdot 3}_{6} + 2 - 3 = 5$

Linear approximation: $z \approx z_0 + \underbrace{f_x(x_0, y_0)(x - x_0)}_{\text{change in } z \text{ caused by } x} + \underbrace{f_y(x_0, y_0)(y - y_0)}_{\text{change in } z \text{ caused by } y}$

$f_x = 1 \cdot y + 1 - 0 = y + 1 \qquad f_y = x \cdot 1 + 0 - 1 = x - 1$

$f_x(x_0, y_0) = f_x(2,3) = 3 + 1 = 4$

$f_y(x_0, y_0) = f_y(2,3) = 2 - 1 = 1$

Lin. approx: $z \approx 5 + 4(x - 2) + 1(y - 3)$

OR

$5 + 4x - 8 + y - 3$

OR

$4x + y - 6$

So $f(2.03, 2.99) \approx 5 + 4(2.03 - 2) + 1(2.99 - 3)$

$= 5 + 4(0.03) + 1(-0.01)$

$= 5 + 0.12 - 0.01$

$= 5.11$

Question 4. Find all critical points of the function $f(x, y) = x^4 + 2y^2 - 4xy$.

$$f_x = 4x^3 + 0 - 4 \cdot 1 \cdot y = 4x^3 - 4y$$

$$f_y = 0 + 4y - 4x \cdot 1 = 4y - 4x$$

Critical points satisfy <u>both</u> equations (i) $4x^3 - 4y = 0$

(ii) $4y - 4x = 0$

$\Rightarrow$ $x^3 - y = 0$

$y - x = 0$ $\Rightarrow y = x$ $\Rightarrow$ plug into first eqn: $x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$x = 0, \; x = -1, \; x = 1$

<u>AND</u> $y = x$

The critical points are $(0, 0)$

$(-1, -1)$

$(1, 1)$