WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #8

Due Tuesday March 12th, in Canvas

Question 1. Find all local maxima, local minima, and saddle points of the function.

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$$

$f_x = 3x^2 + 6x \qquad f_y = 3y^2 - 6y$

Critical points: $3x^2+6x=0 \Rightarrow x^2+2x=0 \Rightarrow x(x+2)=0$

$3y^2-6y=0 \Rightarrow y^2-2y=0 \Rightarrow y(y-2)=0$

$x=0$ or -2, and $y=0$ or 2. FOUR critical points:

(i) $(0,0)$ (ii) $(0,2)$ (iii) $(-2,0)$ (iv) $(-2,2)$

$f_{xx} = 6x+6 \quad f_{xy}=0 \quad f_{yx}=0 \quad f_{yy}=6y-6$

(i) At $(0,0)$ we have $\begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & -6 \end{vmatrix} = -36<0$ SADDLE POINT

(ii) At $(0,2)$ we have $\begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36>0$ $f_{xx}=6>0$ LOCAL MINIMUM

(iii) At $(-2,0)$ we have $\begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36>0$ $f_{xx}=-6<0$ LOCAL MAXIMUM

(iv) At $(-2,2)$ we have $\begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = -36<0$ SADDLE POINT

Question 2. Evaluate the integral

$$\iint_R xy \sin x^2 \, dA$$

where R is the region defined by $0 \le x \le \sqrt{\pi/2}$ and $0 \le y \le 1$.

$$\int_0^{\sqrt{\pi/2}} \int_0^1 xy \sin x^2 \, dy \, dx = \int_0^{\sqrt{\pi/2}} \left[x \frac{y^2}{2} \sin x^2 \right]_{y=0}^{y=1} dx$$

$$= \int_0^{\sqrt{\pi/2}} \frac{1}{2} x \sin x^2 \, dx$$

Sub $u = x^2 \Rightarrow du = 2x\,dx$

$\underbrace{\frac{1}{2} du}_{} = \underbrace{x\,dx}_{}$

$x = 0 \Rightarrow u = 0$

$x = \sqrt{\pi/2} \Rightarrow u = \pi/2$

$$\int_{x=0}^{x=\sqrt{\pi/2}} \frac{1}{2} \sin x^2 \cdot \underbrace{x\,dx}_{} = \int_{u=0}^{u=\pi/2} \frac{1}{2} \sin u \cdot \underbrace{\frac{1}{2} du}_{}$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin u \, du = \frac{1}{4} \left[-\cos u \right]_0^{\pi/2}$$

$$= \frac{1}{4} \left[\cos u \right]_{\pi/2}^{0} = \frac{1}{4} \left(\underbrace{\cos 0}_{1} - \underbrace{\cos \frac{\pi}{2}}_{0} \right) = \frac{1}{4}$$

Question 3. Evaluate the integral.

$$\int_0^{\ln 2} \int_{e^y}^{2} \frac{y}{x}\, dx\, dy$$

$$\int_0^{\ln 2} \left(\int_{x=e^y}^{x=2} y \cdot \frac{1}{x}\, dx \right) dy$$

$$= \int_0^{\ln 2} \Big[y \cdot \ln|x| \Big]_{x=e^y}^{x=2} dy = \int_0^{\ln 2} \big(y\ln 2 - y\underbrace{\ln(e^y)}_{y} \big)\, dy$$

$$= \int_0^{\ln 2} \left(y\ln 2 - y^2 \right) dy = \left[\frac{y^2}{2}\ln 2 - \frac{y^3}{3} \right]_0^{\ln 2}$$

$$= \frac{(\ln 2)^2}{2}\ln 2 - \frac{(\ln 2)^3}{3} = (\ln 2)^3 \cdot \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{(\ln 2)^3}{6}$$