WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #10

Due THURSDAY March 28th, in Canvas

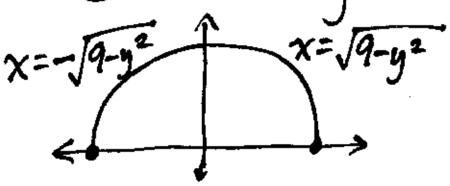
Question 1. Evaluate the integral by converting to cylindrical coordinates. Polar with z

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dz\,dx\,dy$$

Bounds on x are $x = \pm\sqrt{9-y^2} \Rightarrow x^2 = 9-y^2 \Rightarrow x^2+y^2=9$ (circle)

Bounds on y are $0 \le y \le 3 \Rightarrow$ using only top half of circle



In cylindrical coords, $0 \le r \le 3$ and $0 \le \theta \le \pi$.

Bounds on z are $0 \le z \le 9-3\sqrt{x^2+y^2} \Rightarrow 0 \le z \le 9-3r$.

Also remember $dV = r\,dz\,dr\,d\theta$.

$$\text{Integral} = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}\int_{z=0}^{z=9-3r} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}\Big[rz\Big]_{z=0}^{z=9-3r} dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3}(9r-3r^2)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\left[\frac{9r^2}{2} - r^3\right]_{r=0}^{r=3} d\theta = \int_{\theta=0}^{\theta=\pi}\Big(\frac{81}{2} - \underbrace{27}_{54/2}\Big)\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\frac{27}{2}\,d\theta = \pi\cdot\frac{27}{2} = \frac{27\pi}{2}$$

Question 2. Evaluate the integral using spherical coordinates.

$$\iiint_D (x^2+y^2+z^2)^{5/2}\, dV$$

where D is the interior of the unit sphere centered at the origin.

In spherical coordinates, we have $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$ and $x^2+y^2+z^2 = \rho^2$.

Interior of unit sphere centered at origin: $0 \le \rho \le 1$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1} \underbrace{(\rho^2)^{5/2}}_{\rho^5} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^7 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^8}{8} \sin\varphi \right]_{\rho=0}^{\rho=1} d\varphi \, d\theta$$

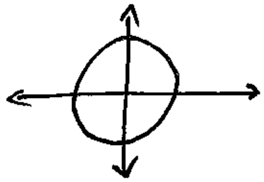
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{8}\sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \left[-\frac{1}{8}\cos\varphi \right]_{\varphi=0}^{\varphi=\pi} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8}\cos\varphi \right]_{\varphi=\pi}^{\varphi=0} d\theta = \int_0^{2\pi} \underbrace{\left(\frac{1}{8} - \frac{-1}{8} \right)}_{\frac{1}{4}} d\theta$$

$$= 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

Question 3. Find the volume bounded by $x^2 + y^2 = 1$, $z = 0$, and $y + z = 1$.

$x^2+y^2=1$:



Can use cylindrical coordinates

$0 \le r \le 1$, $0 \le \theta \le 2\pi$

"Bottom surface" $z=0$, "top surface" $z = 1-y = 1 - r\sin\theta$

$dV = r\,dz\,dr\,d\theta$

$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=1-r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \Big[rz\Big]_{z=0}^{z=1-r\sin\theta} dr\,d\theta = \int_0^{2\pi}\int_0^1 (r - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\sin\theta\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{3}\sin\theta\right)d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = \frac{1}{2}\Big[\theta\Big]_0^{2\pi} + \frac{1}{3}\underbrace{\Big[\cos\theta\Big]_0^{2\pi}}_{0}$$

$$= \frac{1}{2}\cdot 2\pi = \pi$$