WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #11

Due Tuesday April 2nd, in Canvas

Question 1. Evaluate the line integral

$$\int_C (x^2 + y^2)\, ds$$

where C is the circle of radius 4 centered at the origin.

We can parametrize C as $\vec{r}(t) = (x, y) = (4\cos t, 4\sin t)$ for $0 \le t \le 2\pi$.

Then $\vec{r}\,'(t) = (-4\sin t, 4\cos t)$ so $ds = |\vec{r}\,'(t)|\, dt = \sqrt{\underbrace{16\sin^2 t + 16\cos^2 t}_{16(\sin^2 t + \cos^2 t)}}\, dt$

$= \sqrt{16}\, dt = 4\, dt.$

$$\int_C (x^2 + y^2)\, ds = \int_{t=0}^{t=2\pi} \Big((\underbrace{4\cos t}_{x})^2 + (\underbrace{4\sin t}_{y})^2 \Big) \cdot \underbrace{4\, dt}_{ds}$$

$$= \int_0^{2\pi} \left(16\cos^2 t + 16\sin^2 t\right) \cdot 4\, dt = 16 \cdot 4 \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt$$

$$= 64 \int_0^{2\pi} 1\, dt = 64 \cdot 2\pi = 128\pi$$

Question 2. Suppose C is the circle $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$, and $\mathbf{F} = \langle 1, x \rangle$. Evaluate both of the following integrals.

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \int_C \mathbf{F} \cdot \mathbf{n}\, ds$$

$$\vec{r}(t) = \langle \cos t, \sin t \rangle \Rightarrow x = \cos t, \quad y = \sin t$$

$$dx = -\sin t\, dt, \quad dy = \cos t\, dt$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C \langle 1, x \rangle \cdot d\vec{r}$$

$$= \int_C 1\, dx + x\, dy = \int_{t=0}^{t=2\pi} 1 \cdot (-\sin t)\, dt + \cos t \cdot \cos t\, dy$$

$$= \int_0^{2\pi} (-\sin t + \cos^2 t)\, dt = \underbrace{\int_0^{2\pi} -\sin t\, dt}_{0} + \underbrace{\int_0^{2\pi} \cos^2 t\, dt}_{\pi} = \pi$$

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_C 1\, dy - x\, dx = \int_C 1 \cos t\, dt - \cos t \cdot (-\sin t)\, dt$$

$$= \int_0^{2\pi} (\cos t + \sin t \cos t)\, dt = \int_0^{2\pi} \cos t\, dt + \int_0^{2\pi} \sin t \cos t\, dt$$

$$= \Big[\sin t \Big]_0^{2\pi} + \left[\frac{\sin^2 t}{2} \right]_0^{2\pi} = 0$$

Question 3. Evaluate the integral

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathbf{F} = \langle x, y \rangle$ and C is the parabola $\mathbf{r}(t) = \langle 4t, t^2 \rangle$ for $0 \le t \le 1$.

$$\vec{r}(t) = \langle 4t, t^2 \rangle \Rightarrow x = 4t, \quad y = t^2$$

$$dx = 4dt, \quad dy = 2t\,dt$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C \langle x, y \rangle \cdot d\vec{r}$$

$$= \int_C x\,dx + y\,dy = \int_{t=0}^{t=1} 4t \cdot 4dt + t^2 \cdot 2t\,dt$$

$$= \int_0^1 (16t + 2t^3)\,dt = \left[8t^2 + \frac{2t^4}{4} \right]_0^1$$

$$= 8 + \frac{1}{2} = \frac{17}{2}$$