WRITE YOUR NAME:

MAC 2313 B51 Spring 2024

Written homework #12

Due Tuesday April 9th, in Canvas

Question 1. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y) = (2x, 2y)$$

and let C be the curve parametrized by

$$\mathbf{r}(t) = (x, y) = (t, t^2) \qquad \text{for } 0 \leq t \leq 3.$$

(i) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly.

(ii) Find a potential function for $\mathbf{F}$ and use that to evaluate the integral from part (i) using the fundamental theorem for line integrals.

(i) $\vec{r} = (t, t^2) \Rightarrow d\vec{r} = (1, 2t)\,dt$

On C we have $\vec{F} = (2x, 2y) = (2t, 2t^2)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=3} (2t, 2t^2) \cdot (1, 2t)\,dt = \int_0^3 (2t + 4t^3)\,dt$$

$$= \left[t^2 + t^4\right]_0^3 = 3^2 + 3^4 = 9 + 81 = 90.$$

(ii) With trial and error, we can find that $f = x^2 + y^2$ is a potential fcn for $\vec{F}$, because $f_x = 2x$ and $f_y = 2y$.

Starting point of C is $(x, y) = (0, 0^2) = (0, 0)$

Ending point of C is $(x, y) = (3, 3^2) = (3, 9)$

Hence value of integral is $f(3, 9) - f(0, 0) = (3^2 + 9^2) - (0^2 + 0^2)$

$$= 9 + 81 = 90.$$

Question 2. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x,y) = (f,g) = (x+y, y-x)$$

and let C be the unit circle parametrized in the usual way by

$$\mathbf{r}(t) = (x,y) = (\cos t, \sin t) \qquad \text{for } 0 \le t \le 2\pi.$$

(i) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly.

(ii) Compute the double integral

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy$$

where R is the interior of C, and verify that you get the same answer as (i), as is consistent with the circulation version of Green's Theorem. (You are allowed to use geometry here if it helps you.)

(i) $\vec{r} = (\cos t, \sin t) \Rightarrow d\vec{r} = (-\sin t, \cos t)dt$

On C we have $\vec{F} = (x+y, y-x) = (\cos t + \sin t, \sin t - \cos t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2\pi} (\cos t + \sin t, \sin t - \cos t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} \left(-\sin t \cos t - \sin^2 t + \sin t \cos t - \cos^2 t\right) dt = \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t\right) dt$$

$$= \int_0^{2\pi} (-1)\,dt = 2\pi \cdot (-1) = -2\pi$$

(ii) $\dfrac{\partial g}{\partial x} = 0 - 1 = -1 \qquad \dfrac{\partial f}{\partial y} = 0 + 1 = 1$

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy = \iint_R (-1-1)\,dx\,dy = \iint_R (-2)\,dx\,dy$$

Shortcut: Integrating a CONSTANT over R gives that constant times area of R

R is the unit disk with area $\pi 1^2 = \pi$ so integral is $-2 \cdot \pi$