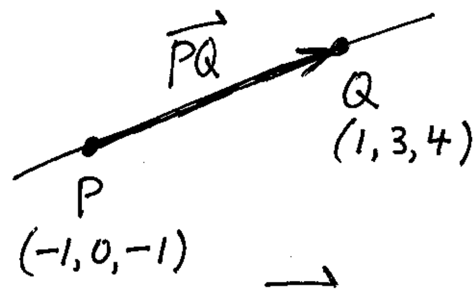
WRITE YOUR NAME:

MAC 2313 Test 1 Thursday February 8th
Total possible score: 18 points

Question 1. Find an equation for the line passing through the points $\underbrace{(-1, 0, -1)}_{P}$ and $\underbrace{(1, 3, 4)}_{Q}$.



A direction vector for the line is $\overrightarrow{PQ} = Q - P$
$= (1,3,4) - (-1,0,-1) = (2,3,5)$

An equation for the line is

$$\boxed{(x,y,z) = (-1,0,-1) + t(2,3,5)}$$

or
$$x = -1 + 2t$$
$$y = 3t$$
$$z = -1 + 5t$$

or $\vec{r}(t) = \langle -1 + 2t, 3t, -1 + 5t \rangle$

Question 2. Find the point where the line $(x, y, z) = (1 + 2t, 2 + 2t, 3 + t)$ intersects the plane $x + y - z = 15$.

Points on the line satisfy $x = 1+2t$, $y = 2+2t$, $z = 3+t$.
Substitute those into the equation of the plane.

$$\underbrace{(1+2t)}_{x} + \underbrace{(2+2t)}_{y} - \underbrace{(3+t)}_{z} = 15$$

$$1+2t+2+2t-3-t = 15$$

$$3t = 15$$

$$t = 5$$

The point is $(x, y, z) = (\underbrace{1+2\cdot 5}_{1+10}, \underbrace{2+2\cdot 5}_{2+10}, 3+5)$

$$= (11, 12, 8).$$

Question 3. Find the angle between the vectors $\mathbf{u} = \langle 1, 0, 1\rangle$ and $\mathbf{v} = \langle 2, 2, 1\rangle$.

Use $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$

$$\Rightarrow \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \cos\theta$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 0 \cdot 2 + 1 \cdot 1 = 2 + 0 + 1 = 3$$

$$|\vec{u}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$$

$$|\vec{v}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{3}{\sqrt{2} \cdot 3} = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{4}} \text{ or } 45^\circ$$

Remember

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Question 4. Find a **unit** vector that is perpendicular to both of the vectors $\mathbf{u} = \langle 2, 0, -1 \rangle$ and $\mathbf{v} = \langle 0, 1, -1 \rangle$.

We can get a vector perpendicular to both given vectors using cross product.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0-(-1), 0-(-2), 2-0 \rangle = \langle 1, 2, 2 \rangle$$

To get a unit vector, divide by its length.

$$\sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\boxed{\text{Answer: } \frac{1}{3}\langle 1,2,2 \rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle}$$

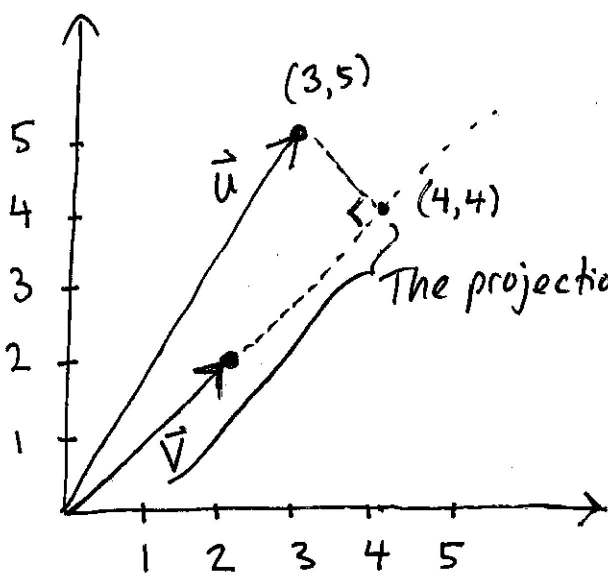
Question 5. Let $\mathbf{u} = \langle 3,5 \rangle$ and let $\mathbf{v} = \langle 2,2 \rangle$. Calculate the projection of $\mathbf{u}$ onto $\mathbf{v}$. Also draw a picture.

$$\mathrm{proj}_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\, \vec{v}$$

$$\vec{u} \cdot \vec{v} = \langle 3,5 \rangle \cdot \langle 2,2 \rangle = 3 \cdot 2 + 5 \cdot 2 = 6 + 10 = 16$$

$$\vec{v} \cdot \vec{v} = \langle 2,2 \rangle \cdot \langle 2,2 \rangle = 2 \cdot 2 + 2 \cdot 2 = 4 + 4 = 8$$

$$\mathrm{proj}_{\vec{v}}\, \vec{u} = \frac{16}{8} \langle 2,2 \rangle = 2 \langle 2,2 \rangle = \langle 4,4 \rangle$$



The projection of $\vec{u}$ onto $\vec{v}$ is $\langle 4,4 \rangle$

Question 6. Find the equation of the plane that is parallel to the vectors $\underbrace{\langle 0,3,2\rangle}_{\vec{u}}$ and $\underbrace{\langle 2,0,1\rangle}_{\vec{v}}$ and passes through the point $(5,0,2)$.

If both of the vectors $\vec{u}$ and $\vec{v}$ are parallel to the plane then $\vec{n} = \vec{u} \times \vec{v}$ will be perpendicular to the plane and if we have a perpendicular/normal vector, then we almost have an equation of the plane. ($Ax+By+Cz=D \Leftrightarrow$ normal vector is $\langle A,B,C\rangle$)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \langle 3-0, 4-0, 0-6\rangle = \langle 3,4,-6\rangle$$

Equation of the plane will have the form $3x+4y-6z=d$.

Finally, use the known point on the plane to find d.

$$3\cdot\underbrace{5}_{x} + 4\cdot\underbrace{0}_{y} - 6\cdot\underbrace{2}_{z} = d$$

$$15 + 0 - 12 = d$$

$$3 = d$$

Answer: $\boxed{3x+4y-6z=3}$

Also correct: $3(x-5)+4(y-0)-6(z-2)=0$

Question 7. For the curve defined by $\mathbf{r}(t) = \langle \sin 2t, 2\sin^2 t, 2\cos t\rangle$, find the **unit** tangent vector when $t = \pi/2$.

$$\vec{r}(t) = \langle \sin(2t), 2(\sin t)^2, 2\cos t \rangle$$

$$\vec{r}\,'(t) = \langle \cos(2t)\cdot 2, 2\cdot 2\sin t\cdot\cos t, -2\sin t \rangle$$

$$= \langle 2\cos(2t), 4\sin t\cos t, -2\sin t \rangle$$ This is a tangent vector

$$\vec{r}\,'\left(\tfrac{\pi}{2}\right) = \langle 2\underbrace{\cos(\pi)}_{-1}, 4\underbrace{\sin\tfrac{\pi}{2}}_{1}\underbrace{\cos\tfrac{\pi}{2}}_{0}, -2\underbrace{\sin\tfrac{\pi}{2}}_{1} \rangle$$

$$= \langle -2, 0, -2 \rangle$$ This is a tangent vector when $t = \pi/2$

We can get a unit tangent vector by dividing by its length.

$$\sqrt{(-2)^2 + 0^2 + (-2)^2} = \sqrt{4+0+4} = \sqrt{8} = 2\sqrt{2}$$

Answer: $\frac{1}{2\sqrt{2}}\langle -2, 0, -2 \rangle = \langle -\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$

or $\langle -\frac{\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2} \rangle$

Question 8. Find the velocity, speed, and acceleration of a particle whose position in three-dimensional space is given by

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$$

$\underline{\text{Velocity}} = \vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$

$\underline{\text{Speed}} = |\vec{v}(t)| = \sqrt{\underbrace{(-4\sin t)^2 + (4\cos t)^2}_{16\sin^2 t + 16\cos^2 t \, = 16(\underbrace{\sin^2 t + \cos^2 t}_{1})} + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\underline{\text{Acceleration}} = \vec{v}\,'(t) = \langle -4\cos t, -4\sin t, 0 \rangle$

Question 9. Find the length of the curve

$$\mathbf{r}(t) = \langle t^3, 3t^2, 6t \rangle$$

from $t = 0$ to $t = 1$.

$$\vec{r}\,'(t) = \langle 3t^2, 6t, 6 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(3t^2)^2 + (6t)^2 + 6^2} = \sqrt{\underbrace{9t^4 + 36t^2 + 36}}$$

This is a perfect square!

$9t^4 = (3t^2)^2$

$36 = 6^2$

and if we consider $(3t^2+6)^2$ then the middle term is $2\cdot 6\cdot 3t^2 = 36t^2$.

$$= \sqrt{(3t^2+6)^2} = 3t^2 + 6$$

$$\text{Length} = \int_{t=0}^{t=1} |\vec{r}\,'(t)|\,dt = \int_0^1 (3t^2+6)\,dt$$

$$= \left[t^3 + 6t\right]_0^1 = 1^3 + 6\cdot 1 = 1+6 = \boxed{7}$$