WRITE YOUR NAME:

MAC 2313 Test 2 Thursday March 14th
Total possible score: 18 points

Question 1. Prove that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{5x^2+8y^2}$$

Experiment with different ways for (x,y) to approach $(0,0)$

If (x,y) approaches $(0,0)$ along the line $y=0$ then

$$\frac{xy}{5x^2+8y^2} = \frac{x\cdot 0}{5x^2+8\cdot 0^2} = \frac{0}{5x^2} = 0$$

If (x,y) approaches $(0,0)$ along the line $y=x$ then

$$\frac{xy}{5x^2+8y^2} = \frac{x\cdot x}{5x^2+8x^2} = \frac{x^2}{13x^2} = \frac{1}{13}$$

Since two different paths to $(0,0)$ result in the function approaching two different numbers, this shows that the limit does not exist.

Question 2. Find f_x, f_y, and f_z for the following function.

$$f(x,y,z) = \sin(2xy) + \sin(3yz) + \sin(5zx)$$

$$f_x = \underbrace{\cos(2xy)}_{\substack{\text{Took deriv. of outer}\\ \text{Left inner alone}}} \cdot \underbrace{2y}_{\frac{\partial}{\partial x}\text{ of inner}} + 0 + \cos(5zx)\cdot 5z$$

$$f_y = \cos(2xy)\cdot 2x + \cos(3yz)\cdot 3z + 0$$

$$f_z = 0 + \cos(3yz)\cdot 3y + \cos(5zx)\cdot 5x$$

Question 3. Given

$$z = 3x^2 - 2xy + y^2, \qquad x = 3u + 2v, \qquad y = 4u - v$$

find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = -2$.

$$z = 3x^2 - 2xy + y^2 \quad \nearrow z_x = 6x - 2y \quad \searrow z_y = -2x + 2y$$

$$x = 3u + 2v \quad \nearrow x_u = 3 \quad \searrow x_v = 2 \qquad y = 4u - v \quad \nearrow y_u = 4 \quad \searrow y_v = -1$$

If $u = 1$ and $v = -2$ then $x = 3 \cdot 1 + 2 \cdot (-2) = 3 - 4 = -1$

$y = 4 \cdot 1 - (-2) = 4 + 2 = 6$

and then $z_x = 6 \cdot (-1) - 2 \cdot 6 = -6 - 12 = -18$

$z_y = -2 \cdot (-1) + 2 \cdot 6 = 2 + 12 = 14$

$$\frac{\partial z}{\partial u} = z_x x_u + z_y y_u = -18 \cdot 3 + 14 \cdot 4 = -54 + 56 = 2$$

$$\frac{\partial z}{\partial v} = z_x x_v + z_y y_v = -18 \cdot 2 + 14 \cdot (-1) = -36 - 14 = -50$$

Question 4. Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at the point $(1, -1, 2)$ in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$.

$$f_x = 1 \cdot y + 0 + z \cdot 1 = y + z$$

$$f_y = x \cdot 1 + 1 \cdot z + 0 = x + z$$

$$f_z = 0 + y \cdot 1 + 1 \cdot x = y + x$$

At $(1,-1,2)$ we have
$$\left.\begin{aligned} f_x &= -1 + 2 = 1 \\ f_y &= 1 + 2 = 3 \\ f_z &= -1 + 1 = 0 \end{aligned}\right\} \nabla f = (1, 3, 0)$$

For directional derivative, need unit vector in direction of $3\vec{i} + 6\vec{j} - 2\vec{k}$

$$\sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

Unit vector is $\vec{u} = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}$ or $\left(\frac{3}{7}, \frac{6}{7}, \frac{-2}{7}\right)$

Directional derivative $= \nabla f \cdot \vec{u} = (1, 3, 0) \cdot \left(\frac{3}{7}, \frac{6}{7}, \frac{-2}{7}\right)$

$$= \frac{3}{7} + \frac{18}{7} = \frac{21}{7} = 3$$

Question 5. The point $(2, 3, 0)$ lies on the surface $z = \ln(x^2 + xy - y^2)$. Use tangent planes to estimate the value of z when $x = 2.001$ and $y = 3.002$.

$$f(x,y) = \ln(x^2 + xy - y^2)$$

$$f_x(x,y) = \frac{1}{x^2+xy-y^2} \cdot (2x+y)$$

$$f_y(x,y) = \frac{1}{x^2+xy-y^2} \cdot (x-2y)$$

$$x_0 = 2, \quad y_0 = 3, \quad z_0 = 0$$

$$f_x(x_0,y_0) = \frac{1}{\underbrace{2^2+2\cdot 3-3^2}_{4+6-9=1}} \cdot (2\cdot 2+3) = 7$$

$$f_y(x_0,y_0) = \frac{1}{2^2+2\cdot 3-3^2} \cdot (2-2\cdot 3) = -4$$

$$z \approx z_0 + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$$

If $x = 2.001$ and $y = 3.002$ then

$$z \approx 0 + 7\cdot(0.001) + (-4)\cdot(0.002)$$

$$= 0 + 0.007 - 0.008$$

$$= -0.001$$

Question 6. Find all local maxima, local minima, and saddle points of the function.

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$$

$f_x = 3x^2 + 6x \qquad f_y = 3y^2 - 6y$

$f_{xx} = 6x + 6 \qquad f_{yx} = 0$

$f_{xy} = 0 \qquad f_{yy} = 6y - 6$

Critical points: $f_x = 0$ and $f_y = 0$

$3x^2 + 6x = 0 \Rightarrow x^2 + 2x = 0 \Rightarrow x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$

$3y^2 - 6y = 0 \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$

Four critical points: $(0,0)$ $(0,2)$ $(-2,0)$ $(-2,2)$

$(0,0)$ $D = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & -6 \end{vmatrix} = -36 < 0$

SADDLE POINT

$(0,2)$ $D = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 > 0 \quad f_{xx} = 6 > 0$

LOCAL MIN

$(-2,0)$ $D = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 > 0 \quad f_{xx} = -6 < 0$

LOCAL MAX

$(-2,2)$ $D = \begin{vmatrix} 6x+6 & 0 \\ 0 & 6y-6 \end{vmatrix} = \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = -36 < 0$

SADDLE POINT

Question 7. Evaluate the integral.

$$\int_1^{49}\int_0^{15}(x+y)^{-3/2}\,dx\,dy$$

$$\int_1^{49}\Bigg(\underbrace{\int_{x=0}^{x=15}(x+y)^{-3/2}\,dx}_{\substack{x \text{ is the variable} \\ y \text{ is constant}}}\Bigg)\,dy$$

$$= \int_1^{49}\left[\frac{(x+y)^{-1/2}}{-1/2}\right]_{x=0}^{x=15} dy = -2\int_1^{49}\left((15+y)^{-1/2} - y^{-1/2}\right)dy$$

$$= -2\left[\frac{(15+y)^{1/2}}{1/2} - \frac{y^{1/2}}{1/2}\right]_1^{49} = -4\left[(15+y)^{1/2} - y^{1/2}\right]_1^{49}$$

$$= -4\left[(15+y)^{1/2}\right]_1^{49} + 4\left[y^{1/2}\right]_1^{49}$$

$$= -4\left(64^{1/2} - 16^{1/2}\right) + 4\left(49^{1/2} - 1^{1/2}\right)$$

$$= -4(8-4) + 4(7-1) = -4\cdot 4 + 4\cdot 6$$

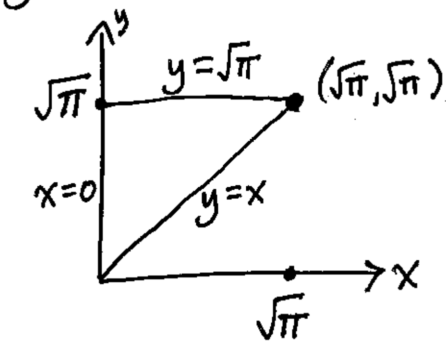
$$= 8$$

Question 8. Evaluate the integral by changing the order of integration.

$$\int_0^{\sqrt{\pi}} \int_x^{\sqrt{\pi}} y^2 \cos(xy)\, dy\, dx$$

Currently written as Type 1. $0 \le x \le \sqrt{\pi}$. $y = x$ is bottom curve
$y = \sqrt{\pi}$ is top curve

Region is a triangle:



If we rewrite as Type 2 then left curve is $x = 0$ $\quad 0 \le x \le y$
right curve is $x = y$ $\quad 0 \le y \le \sqrt{\pi}$

$$\int_0^{\sqrt{\pi}} \Bigg(\underbrace{\int_{x=0}^{x=y} y^2 \cos(xy)\, dx}_{x \text{ is the variable, } y \text{ is constant. Like integrating } 100\cos(10x).} \Bigg) dy$$

$$= \int_0^{\sqrt{\pi}} \Big[y \sin(xy) \Big]_{x=0}^{x=y} dy = \int_0^{\sqrt{\pi}} y \sin(y^2)\, dy \qquad \{\sin 0 = 0\}$$

Now sub $u = y^2 \Rightarrow du = 2y\,dy \Rightarrow \frac{1}{2} du = y\,dy$. $\quad y = 0 \Rightarrow u = 0^2 = 0$
$y = \sqrt{\pi} \Rightarrow u = (\sqrt{\pi})^2 = \pi$

$$\int_{y=0}^{y=\sqrt{\pi}} \sin(y^2) \cdot y\,dy = \int_{u=0}^{u=\pi} \sin(u) \cdot \frac{1}{2}\, du = \frac{1}{2} \int_0^{\pi} \sin u\, du$$

$$= \frac{1}{2} \Big[-\cos u \Big]_0^{\pi} = \frac{1}{2} \Big[\cos u \Big]_{\pi}^{0} = \frac{1}{2} \big(\underbrace{\cos 0}_{1} - \underbrace{\cos \pi}_{-1} \big) = \frac{1}{2} \big(1 - (-1) \big)$$

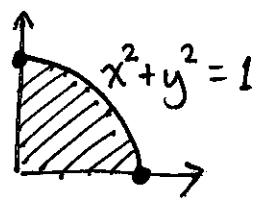
$$= \frac{1}{2}(2) = 1$$

Question 9. Find the average value of the function $f(x,y) = xy$ on the region defined by $x^2 + y^2 \leq 1$, $x \geq 0$, and $y \geq 0$. Call this region R

Avg value of f on R is $\dfrac{\iint_R f(x,y)\,dA}{\text{area of } R}$

Area of R is $\iint_R 1\,dA$ but in some cases, we can find the area by knowing geometry

Picture of R:



One quarter of unit disk

Area of R is $\frac{1}{4}\pi \cdot 1^2 = \frac{\pi}{4}$

Can write as Type 1 region: $0 \leq y \leq \sqrt{1-x^2}$ and $0 \leq x \leq 1$

$$\iint_R xy\,dA = \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x(1-x^2)\,dx = \frac{1}{2}\int_0^1 (x - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$$

So average value $= \frac{1}{8} \div \frac{\pi}{4} = \frac{1}{8}\cdot\frac{4}{\pi} = \dfrac{1}{2\pi}$