WRITE YOUR NAME:

MAC 2313 Test 3 Thursday April 11th
Total possible score: 18 points

Question 1. Evaluate the integral using any correct method.

$$\int_0^5 \int_0^{\sqrt{25-y^2}} y\,dx\,dy$$

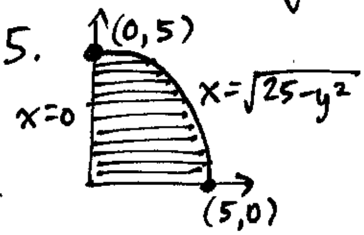
METHOD 1. Rectangular. $\displaystyle\int_0^5 \left(\int_{x=0}^{x=\sqrt{25-y^2}} y\,dx\right) dy$

$$= \int_0^5 \left[xy\right]_{x=0}^{x=\sqrt{25-y^2}} dy = \int_0^5 y\sqrt{25-y^2}\,dy$$

Sub $u = 25-y^2$
$du = -2y\,dy$
$-\frac{1}{2}du = y\,dy$

$$\int_{u=25}^{u=0} -\frac{1}{2}u^{1/2}\,du = \frac{1}{2}\int_0^{25} u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^{25} = \frac{1}{3}\cdot 5^3 = \frac{125}{3}$$

METHOD 2. Polar. $0 \le x \le \sqrt{25-y^2} \Rightarrow$ left curve $x=0$, right curve $x^2+y^2=25$
Also $0 \le y \le 5$.



Polar: $0 \le r \le 5$, $0 \le \theta \le \frac{\pi}{2}$
$y = r\sin\theta$ $\quad dx\,dy = dA = r\,dr\,d\theta$

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=5} \underbrace{r\sin\theta}_{y}\cdot\underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\pi/2}\int_0^5 r^2\sin\theta\,dr\,d\theta$$

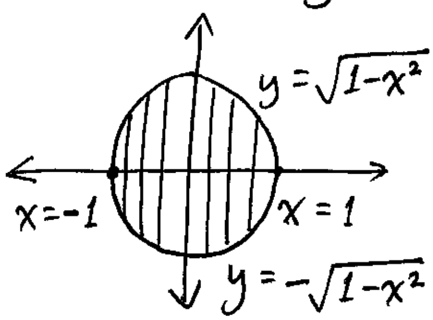
$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\sin\theta\right]_0^5 d\theta = \int_0^{\pi/2}\frac{125}{3}\sin\theta\,d\theta = \left[-\frac{125}{3}\cos\theta\right]_0^{\pi/2}$$

$$= 0 - \left(-\frac{125}{3}\right) = \frac{125}{3}$$

Question 2. Evaluate the integral by converting to polar coordinates.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{x^2+y^2+1}\,dy\,dx$$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \Rightarrow$ top and bottom curves satisfy $x^2+y^2=1$



Polar: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$dy\,dx = dA = r\,dr\,d\theta$

$x^2+y^2=r^2$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} \frac{1}{r^2+1}\, r\,dr\,d\theta = \int_0^{2\pi}\left(\int_0^1 \frac{r}{r^2+1}\,dr\right)d\theta$$

Sub $u=r^2+1$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$$= \int_0^{2\pi}\Bigg(\underbrace{\int_{u=1}^{u=2} \frac{1}{2}\cdot\frac{1}{u}\,du}_{\frac{1}{2}\left[\ln|u|\right]_1^2}\Bigg)d\theta = \int_0^{2\pi}\left(\frac{1}{2}\ln 2\right)d\theta$$

$$= 2\pi\cdot\frac{1}{2}\ln 2$$

$$= \boxed{\pi \ln 2}$$

Question 3. Evaluate the integral.

$$\int_0^1 \int_0^{3-x} \int_0^{1+x+2y} dz\, dy\, dx$$

$$\int_0^1 \int_0^{3-x} \Big[z \Big]_{z=0}^{z=1+x+2y} dy\, dx$$

$$= \int_0^1 \int_0^{3-x} (1+x+2y)\, dy\, dx$$

$$= \int_0^1 \Big[y + xy + y^2 \Big]_{y=0}^{y=3-x} dx$$

$$= \int_0^1 \Big(3-x + \underbrace{x(3-x)}_{3x-x^2} + \underbrace{(3-x)^2}_{9-6x+x^2} \Big) dx$$

$$= \int_0^1 (12-4x)\, dx = \Big[12x - 2x^2 \Big]_0^1$$

$$= 12 - 2 = 10$$

Question 4. Evaluate the integral.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/2}\left(\int_{\rho=0}^{\rho=1} \rho^2 \sin^3\phi \, d\rho\right) d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\rho^3}{3}\sin^3\phi\right]_{\rho=0}^{\rho=1} d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{3}\sin^3\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\left(\int_{\phi=0}^{\phi=\pi/2} \frac{1}{3}\underbrace{\sin^2\phi}_{1-\cos^2\phi} \sin\phi \, d\phi\right) d\theta$$

Sub $u = \cos\phi$
$du = -\sin\phi \, d\phi$

$$= \int_0^{2\pi}\left(\int_{u=1}^{u=0} \frac{1}{3}(1-u^2)(-du)\right) d\theta = \int_0^{2\pi}\left(\int_0^1 \frac{1}{3}(1-u^2)\,du\right) d\theta$$

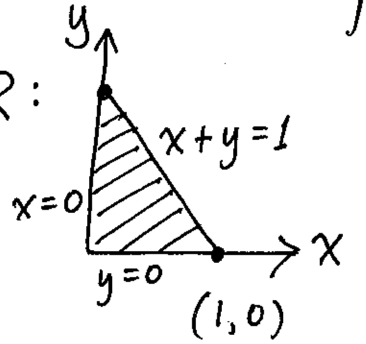
$$= \int_0^{2\pi}\left[\frac{1}{3}\left(u - \frac{u^3}{3}\right)\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{3}\cdot\frac{2}{3}\, d\theta$$

$$= 2\pi \cdot \frac{2}{9} = \boxed{\frac{4\pi}{9}}$$

Question 5. Evaluate the integral, where R is the region bounded by $x = 0$, $y = 0$, and $x + y = 1$.

$$\iint_R ((x+y)^2 - (x-y)^2)\, dA$$

R:



$$\iint_R \left(x^2 + 2xy + y^2 \underbrace{- (x^2 - 2xy + y^2)}_{= -x^2 + 2xy - y^2}\right) dA$$

$$= \iint_R 4xy\, dA$$

Could write R in terms of a "top curve" and "bottom curve"

Top: $y = 1 - x$ Bottom: $y = 0$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 4xy\, dy\, dx$$

$$= \int_{x=0}^{x=1} \Big[2xy^2 \Big]_{y=0}^{y=1-x} dx = \int_0^1 2x(1-x)^2\, dx$$

$$= \int_0^1 2x(1 - 2x + x^2)\, dx = \int_0^1 (2x - 4x^2 + 2x^3)\, dx$$

$$= \left[x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \right]_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{6}{6} - \frac{8}{6} + \frac{3}{6} = \boxed{\frac{1}{6}}$$

Question 6. Evaluate the integral, where C is the straight line segment from $(1,2,3)$ to $(3,6,7)$.

$$\int_C (x+y+z)\,ds$$

Let $P=(1,2,3)$ and $Q=(3,6,7)$

Vector from P to Q is $(3,6,7)-(1,2,3)=(2,4,4)$

Line segment from P to Q can be parametrized as $(1,2,3)+t(2,4,4)$ as t goes from 0 to 1.

$$x = 1+2t \quad \Rightarrow dx = 2dt$$
$$y = 2+4t \quad \Rightarrow dy = 4dt$$
$$z = 3+4t \quad \Rightarrow dz = 4dt$$

$$ds = \sqrt{(dx)^2+(dy)^2+(dz)^2} \text{ or } \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$ds = \sqrt{2^2+4^2+4^2}\,dt = \sqrt{4+16+16}\,dt = \sqrt{36}\,dt = 6dt$$

$$\int_{t=0}^{t=1}\Big((1+2t)+(2+4t)+(3+4t)\Big)\,6dt = \int_0^1 (6+10t)\,6\,dt$$

$$= \int_0^1 (36+60t)\,dt = \Big[36t+30t^2\Big]_0^1 = \boxed{66}$$

Question 7. Evaluate the integral

$$\int_C \underbrace{(ye^x + \sin y)}_{M}\, dx + \underbrace{(e^x + \sin y + x\cos y)}_{N}\, dy$$

where C is the line segment from $(1,0)$ to $(0,\pi/2)$. Use any correct method.

Given integral can be written $\int_C M\,dx + N\,dy = \int_C (M,N)\cdot(dx,dy)$

$= \int \vec{F}\cdot d\vec{r}$ where $\vec{F} = (M,N)$

Could there be a POTENTIAL FUNCTION for $\vec{F}$?

$M = ye^x + \sin y$ — integrate wrt x (y is constant) — $f = ye^x + x\sin y + C_1$ (can depend on y)

$N = e^x + \sin y + x\cos y$ — integrate wrt y (x is constant) — $f = e^x y - \cos y + x\sin y + C_2$ (can depend on x)

A potential function is $f(x,y) = ye^x + x\sin y - \cos y$.

By the Fundamental Theorem for Line Integrals, the integral is equal to

$$f(0,\tfrac{\pi}{2}) - f(1,0) = \Big(\underbrace{\tfrac{\pi}{2}\cdot e^0}_{\frac{\pi}{2}} + \underbrace{0\sin\tfrac{\pi}{2}}_{0} - \underbrace{\cos\tfrac{\pi}{2}}_{0}\Big) - \Big(\underbrace{0\cdot e^1}_{0} + \underbrace{1\sin 0}_{0} - \underbrace{\cos 0}_{1}\Big)$$

$$= \frac{\pi}{2} - (-1) \quad \text{or} \quad \frac{\pi}{2} + 1 \quad \text{or} \quad \frac{\pi+2}{2}$$

Question 8. Calculate the flux of the field $\mathbf{F}(x,y) = (x,y)$ across the closed semicircular path that consists of the semicircular arc $\mathbf{r}(t) = (3\cos t, 3\sin t)$, $0 \le t \le \pi$, followed by the line segment $\mathbf{r}(t) = (t, 0)$, $-3 \le t \le 3$.

Flux of $\vec{F} = (M,N)$ across C is $\int_C M\,dy - N\,dx$

$= \int x\,dy - y\,dx$ in this case.

Let C_1 be the first part: $x = 3\cos t$, $y = 3\sin t$

$dx = -3\sin t\,dt$, $dy = 3\cos t\,dt$

$0 \le t \le \pi$

Let C_2 be the second part: $x = t$, $y = 0$

$dx = dt$, $dy = 0$ $\quad -3 \le t \le 3$

$$\int_{C_1} x\,dy - y\,dx = \int_{t=0}^{t=\pi} 3\cos t \cdot 3\cos t\,dt - 3\sin t \cdot (-3\sin t)\,dt$$

$$= \int_0^\pi (9\cos^2 t + 9\sin^2 t)\,dt = \int_0^\pi 9\,dt = 9\pi$$

$$\int_{C_2} x\,dy - y\,dx = \int_{t=-3}^{t=3} t \cdot 0 - 0 \cdot dt = 0$$

Answer: $\boxed{9\pi}$

Note: Although Green's theorem was not on the list of topics for this test, it can also be used here.

The flux here is also equal to the double integral of $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ (the "divergence") over the interior of the semicircle

Question 9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ and C is the circle with radius 2 centered at the origin.

We can parametrize C as $\vec{r} = (x, y) = (2\cos t, 2\sin t)$, $0 \le t \le 2\pi$.

Then $d\vec{r} = (-2\sin t, 2\cos t)\, dt$.

On C, we have $\vec{F} = \left(\frac{-2\sin t}{\underbrace{4\cos^2 t + 4\sin^2 t}_{4}}, \frac{2\cos t}{\underbrace{4\cos^2 t + 4\sin^2 t}_{4}} \right)$

which simplifies to $\vec{F} = \left(-\frac{1}{2}\sin t, \frac{1}{2}\cos t\right)$.

Then $\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2\pi} \underbrace{\left(-\frac{1}{2}\sin t, \frac{1}{2}\cos t\right)}_{\vec{F}} \bullet \underbrace{(-2\sin t, 2\cos t)\, dt}_{d\vec{r}}$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi \cdot 1 = 2\pi.$