Vectors are often described as quantities with both *magnitude* and *direction*.

Some common examples are velocity, acceleration, and force.

Vectors might lie in two-dimensional space ($\mathbb{R}^2$), three-dimensional space ($\mathbb{R}^3$), or even in higher-dimensional space (but in this course we won't go beyond three dimensions).

We often visualize a vector as an arrow (although a vector can also represent a location). Mathematically, a vector in $\mathbb{R}^2$ is just an ordered pair of real numbers, and a vector in $\mathbb{R}^3$ is just an ordered triple of real numbers.

In our textbook (and many other typed materials), vectors are written in boldface, like $\mathbf{v}$ or $\mathbf{w}$. When writing by hand, we often use an underline, overline, or arrow.

The vector from point P to point Q is written $\overrightarrow{PQ}$ and can be calculated as $Q - P$. (It's the vector you need to add to P to get Q.)

QUESTION: If P is the point $(3, 1)$ and Q is the point $(7, 2)$, find the vector $\overrightarrow{PQ}$. Also draw a picture.

A 'scalar' just means a (single) real number. We can multiply a vector by a scalar and get a parallel vector. (We multiply each component by that scalar.)

QUESTION: If $\mathbf{v} = \langle 4, 3 \rangle$ and $c = 2$, compute $c\mathbf{v}$. Also draw a picture.

There are various ways to visualize addition and subtraction of vectors (but mathematically, we just add or subtract corresponding components).

QUESTION: If $\mathbf{v} = \langle 5, 1 \rangle$ and $\mathbf{w} = \langle 2, 4 \rangle$, compute $\mathbf{v} + \mathbf{w}$. Also draw a picture.

Three-dimensional space is harder to draw than two-dimensional space. When working in $\mathbb{R}^3$, we can think of the xy-plane as the 'floor' and the z-axis as perpendicular to the floor, with positive z values above the floor and negative z values below.

The x-axis, y-axis, and z-axis obey the **right-hand rule**: if you curl the fingers of your right hand from the positive x-axis toward the positive y-axis, your thumb points along the positive z-axis.

How do we find the distance between two points in $\mathbb{R}^2$ or $\mathbb{R}^3$? For instance, what's the distance between $(8, 2, 6)$ and $(3, 5, 7)$ in $\mathbb{R}^3$?

FACT: The distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_3) in $\mathbb{R}^3$ is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

FACT: The **magnitude** (or length) of a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$ is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$$

and the **magnitude** (or length) of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

The only vector whose magnitude is 0 is the zero vector, $\mathbf{0}$.

A vector whose magnitude is 1 is called a **unit vector**. Given any nonzero vector $\mathbf{v}$, we can get a unit vector in the same direction as $\mathbf{v}$ by multiplying by the scalar $1/|\mathbf{v}|$.

QUESTION: Find a unit vector in the same direction as $\mathbf{v} = \langle 3, -2, -6 \rangle$.

Coordinate unit vectors i, j, k

The 'coordinate unit vectors' or 'standard basis vectors' in $\mathbb{R}^2$ are

$$\begin{aligned} \mathbf{i} &= \langle 1, 0 \rangle \\ \mathbf{j} &= \langle 0, 1 \rangle \end{aligned}$$

The 'coordinate unit vectors' or 'standard basis vectors' in $\mathbb{R}^3$ are

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle \\ \mathbf{j} &= \langle 0, 1, 0 \rangle \\ \mathbf{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$

So for example, the vector $\langle 9, -2, 6 \rangle$ can also be written as $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

Properties of vector operations

Vector addition and scalar multiplication obey some nice properties. For example:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} \\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}) \\ \mathbf{v} + \mathbf{0} &= \mathbf{v} \\ 1\mathbf{v} &= \mathbf{v} \\ c(\mathbf{u} + \mathbf{v}) &= c\mathbf{u} + c\mathbf{v} \\ (a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v} \end{aligned}$$

None of these properties is surprising, but they still require proof.

Midpoint formula

The midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Planes parallel to the coordinate planes

In $\mathbb{R}^3$, any equation of the form $x = c$, $y = c$, or $z = c$ (where c is a constant) will define a **plane** (a plane which is parallel to one of the coordinate planes and perpendicular to one of the coordinate axes).

QUESTION: In $\mathbb{R}^3$, which points satisfy $y = 0$? Which points satisfy $y = 5$? Which points satisfy $z = 2$? Draw a picture for each of those three scenarios.

Spheres

What does the equation of a **sphere** look like in three dimensions? For example, how would we find the equation of the sphere whose center is $(7, 1, 2)$ and whose radius is 4?

That sphere is the set of all points (x, y, z) whose distance from $(7, 1, 2)$ is exactly 4.

FACT: The sphere in $\mathbb{R}^3$ whose center is the point (a, b, c) and whose radius is r has the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

(Note: When we say 'sphere', we mean the *surface* of the sphere, like a balloon or soap bubble, but not the air inside of it. If we want to include the interior of the sphere, we use the word 'ball'.)

Dot product

If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are two vectors in $\mathbb{R}^2$, then their dot product is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are two vectors in $\mathbb{R}^3$, then their dot product is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

(In fact, dot product can be defined in a similar way in $\mathbb{R}^n$ for any n.)

Notice that the dot product of two vectors is a **scalar**.

FACT: If $\mathbf{v}$ is any vector, then $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. That is, the dot product of a vector with itself is equal to the square of its length.

The dot product satisfies the following properties, which do not look surprising, but you still need to do a little work to verify that they are true.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
$$c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$$
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

One consequence of these, for example, is that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$.

FACT: If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, then $\mathbf{u}\cdot\mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

EXAMPLE: Consider the vector $\mathbf{u} = \langle 4, 3\rangle$ in $\mathbb{R}^2$. Which vectors are perpendicular to $\mathbf{u}$?

EXAMPLE: Consider the vectors $\mathbf{u} = \langle 1, 3, 2\rangle$ and $\mathbf{v} = \langle 4, -2, 1\rangle$ in $\mathbb{R}^3$. We can check that

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \langle 1,3,2\rangle \cdot \langle 4,-2,1\rangle \\ &= 1\cdot 4 \;+\; 3\cdot(-2) \;+\; 2\cdot 1 \\ &= 4 - 6 + 2 \\ &= 0 \end{aligned}$$

so according to the top of this page, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. But how do we see that?

Let $\mathbf{u}$ and $\mathbf{v}$ be *any* two vectors in $\mathbb{R}^3$, and let θ be the angle between them. Create a triangle by also drawing the vector $\mathbf{u} - \mathbf{v}$, and then consider what the Pythagorean theorem (and its converse) say about this triangle.

Dot products are closely related to **projections**.

If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors, the projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$, is like the 'shadow' that $\mathbf{u}$ casts on $\mathbf{v}$.

The projection $\text{proj}_{\mathbf{v}}\mathbf{u}$ is equal to $k\mathbf{v}$ for some scalar k. The scalar k is positive if the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute, and is negative if the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.

What is k? The key to figuring this out is that $\mathbf{u} - k\mathbf{v}$ is perpendicular to $\mathbf{v}$.

FACT: The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

You could memorize this formula if you want, but rote memorization can be risky. Tips for making sense of the formula: Note that $\mathbf{u}$ appears just once, in the numerator, and $\mathbf{v}$ appears in a 'balanced' way.

The dot product is also related to the **cosine** of the angle between two vectors.

FACT: The dot product of any two vectors $\mathbf{u}$ and $\mathbf{v}$ satisfies

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Why is this true? It can be shown using pictures like when we were considering projections, where the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $k\mathbf{v}$ (and the angle between $\mathbf{u}$ and $\mathbf{v}$ could be acute or obtuse).

If the angle θ is acute, then $\cos\theta$ is positive, and if θ is obtuse, then $\cos\theta$ is negative. We can show that in both cases, we have

$$\cos\theta = \frac{k\,|\mathbf{v}|}{|\mathbf{u}|}.$$

But remember that we know $k = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}$. Plug that in for k and watch what happens.

Dot product is also related to **work**. If a force $\mathbf{F}$ on an object produces a displacement $\mathbf{d}$, then the work done by the force is $\mathbf{F} \cdot \mathbf{d}$.

Informally, the dot product is a measure of 'similarity' between vectors. The dot product is positive if the angle between the vectors is close to 0, and the dot product is negative if the angle between the vectors is close to 180 degrees.