Cross product

Cross product is defined **only in three dimensions**.

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are two vectors in $\mathbb{R}^3$, then their cross product is

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle.$$

However, you *don't* need to memorize those *symbols*. You can perform a *procedure* involving a 3 by 3 determinant.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Informally, there are three 'forward' diagonals and three 'backward' diagonals.

EXAMPLE 1: If $\mathbf{u} = \langle 2, 3, 5 \rangle$ and $\mathbf{v} = \langle 7, 8, 9 \rangle$, compute $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 5 \\ 7 & 8 & 9 \end{vmatrix} = \dots$$

In this case, we get $\mathbf{u} \times \mathbf{v} = \langle 3 \cdot 9 - 5 \cdot 8,\ 5 \cdot 7 - 2 \cdot 9,\ 2 \cdot 8 - 3 \cdot 7 \rangle = \dots = \langle -13, 17, -5 \rangle$.

Note that the cross product of two vectors is another **vector**.

FACT: The cross product $\mathbf{u} \times \mathbf{v}$ is a vector that is **perpendicular** to both $\mathbf{u}$ and $\mathbf{v}$.

Why create the cross product? Suppose we want a vector $\langle x, y, z \rangle$ that is perpendicular to both $\langle 2, 3, 5 \rangle$ and $\langle 7, 8, 9 \rangle$. Using what we know about **dot** products, that means we want

$$2x + 3y + 5z = 0,$$
$$7x + 8y + 9z = 0.$$

What happens if we try to solve this system of equations?

The cross product obeys the **right-hand rule:** if the fingers of your right hand curl from $\mathbf{u}$ toward $\mathbf{v}$, then your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$. Cross product is also related to **torque**.

FACT: The cross product has the following property.

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. The quantity $|\mathbf{u}|\,|\mathbf{v}|\sin\theta$ is also the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

The cross product obeys various algebraic rules. Some of those rules resemble ordinary multiplication of numbers, such as

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$$

but some do NOT, such as

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}).$$

That is, **order matters** when taking the cross product of two vectors.

The cross products of the coordinate unit vectors are as follows:

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{j} \times \mathbf{i} &= -\mathbf{k} \\ \mathbf{k} \times \mathbf{j} &= -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \end{aligned}$$

Compare and contrast the DOT product and CROSS product

The **dot** product is defined in $\mathbb{R}^n$ for any n.
The **cross** product is defined only in $\mathbb{R}^3$.

The **dot** product of two vectors is a **scalar**.
The **cross** product of two vectors is another **vector**.

The **dot** product is related to the **cosine** of the angle between the vectors.
The **cross** product is related to the **sine** of the angle between the vectors.

The **dot** product $\mathbf{u} \cdot \mathbf{v}$ is equal to 0 if and only if $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular**.
The **cross** product $\mathbf{u} \times \mathbf{v}$ is equal to $\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are **parallel**.

Lines in three-dimensional space

Suppose $\mathbf{r}_0$ is a specific point in space, and $\mathbf{v}$ is a specific vector. Then the equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v} \qquad (-\infty < t < \infty)$$

is a **vector equation** for the line through $\mathbf{r}_0$ parallel to $\mathbf{v}$. (If we restrict t, then we get a line *segment*.)

We can break that into components. If $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = \langle a, b, c\rangle$ then we can rewrite:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

which are the **parametric equations** of a line.

EXAMPLE 2: Find parametric equations for the line through the points $P = (-2, 0, 3)$ and $Q = (4, 5, -3)$, and for the line **segment** from P to Q.

Distance from a point to a line

There is a general formula that can be used to find the distance from any point to any line. You can memorize it if you want, but memorization can be risky.

FACT: Suppose ℓ is the line through point P with direction vector $\mathbf{v}$, and suppose Q is some other point. Then the distance from Q to ℓ is

$$\frac{|\mathbf{v} \times \overrightarrow{PQ}|}{|\mathbf{v}|}$$

An alternative to memorizing that formula is to follow a *strategy* for finding the distance. Can we draw a picture? Does the picture suggest a right-angled triangle? Would a particular trig function be useful? Do we know a vector operation related to that trig function?

EXAMPLE 3: Find the distance from the point $Q = (5, 6, 1)$ to the line given by $x = 1 + 3t$, $y = 3 - 4t$, and $z = 1 + t$.

Determining whether lines intersect

To determine if two lines intersect, we solve a system of three equations in two unknowns. Such a system might have no solutions (meaning the lines do not intersect), one solution (meaning the lines intersect in a single point), or infinitely many solutions (meaning the lines are actually the same line).

Also, if two lines in $\mathbb{R}^3$ do not intersect, that could mean they are parallel (which means there is a plane containing both of them) or they could be nonparallel lines which can be contained in two parallel planes (these are called 'skew lines').

EXAMPLE 4a: Determine whether the lines intersect (and describe the way in which they do or do not intersect).

$$\ell_1 = (1,3,2) + t\langle 6,-7,1\rangle, \qquad \ell_2 = (10,6,14) + s\langle 3,1,4\rangle$$

EXAMPLE 4b: Determine whether the lines intersect (and describe the way in which they do or do not intersect).

$$\ell_1 = (4,6,1) + t\langle 0,-1,1\rangle, \qquad \ell_2 = (-3,1,4) + s\langle -7,4,-1\rangle$$

EXAMPLE 4c: Determine whether the lines intersect (and describe the way in which they do or do not intersect).

$$\ell_1 = (1,7,6) + t\langle 2,-3,1\rangle, \qquad \ell_2 = (-9,22,1) + s\langle 6,-9,3\rangle$$

EXAMPLE 4d: Determine whether the lines intersect (and describe the way in which they do or do not intersect).

$$\ell_1 = (4,0,1) + t\langle 1,-2,3\rangle, \qquad \ell_2 = (1,6,4) + s\langle -7,14,-21\rangle$$

Equations of planes

We can describe a plane if we know a point in the plane as well as a **normal** vector to the plane (i.e., a vector that is perpendicular to the plane).

If the point $P_0 = (x_0, y_0, z_0)$ is in the plane and the vector $\mathbf{n} = \langle a, b, c\rangle$ is normal to the plane, then the points in the plane are all the points $P = (x, y, z)$ with the property that the vector from P_0 to P is perpendicular to $\mathbf{n}$. Equivalently,

$$\langle a, b, c\rangle \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which can be written in the form $ax + by + cz = d$ where a, b, c, d are constants. Moreover, any equation of the form $ax + by + cz = d$ describes a plane in $\mathbb{R}^3$.

EXAMPLE 5: Find the equation of the plane passing through the points $(1, 0, 3)$, $(0, 4, 2)$, and $(1, 1, 1)$.

Planes intersecting planes, lines intersecting planes

Two planes might be parallel, in which case the planes have the same normal vector (more precisely, parallel normal vectors), or two planes might intersect in a line, in which case the two planes have nonparallel normal vectors, and the angle between the planes is equal to the angle between their normal vectors.

EXAMPLE 6: Find an equation of the line of intersection of the two planes. Also find the angle between the two planes.

$$-x + 2y + z = 1, \qquad x + y + z = 0$$

EXAMPLE 7: Determine whether the given line and the given plane intersect. If they do, find the point of intersection.

$$(x, y, z) = (5 - 2t, -5 + 3t, -6 + 4t), \qquad 3x + 2y - 4z = -3$$

Distance from a point to a plane

Just as with the distance with a point to a *line*, there is a formula we could memorize, but that can be risky.

FACT: Suppose P is a point in the plane $ax + by + cz = d$ (which means $\langle a, b, c\rangle$ is a normal vector to the plane) and suppose Q is some other point. Then the distance from Q to the plane is

$$\frac{\left|\mathbf{n} \cdot \overrightarrow{PQ}\right|}{|\mathbf{n}|}$$

Just as with the distance from a point to a line, it might be better in some ways to find a *strategy* for finding the distance. Can we draw a picture? Does the picture suggest a right-angled triangle? Would a particular trig function be useful? Do we know a vector operation related to that trig function?

EXAMPLE 8: Find the distance from the point $Q = (1, 2, -4)$ to the plane $2x - 5z = 5$.

Cylinders

In everyday life, the word 'cylinder' usually means the shape that mathematicians call a *circular* cylinder. You can think of that shape as consisting of many circles stacked on top of each other (or alternatively, starting with a circle in a plane and moving it orthogonally to that plane).

In mathematics, the word **cylinder** is more general than a circular cylinder. One way to describe a cylinder is a surface built out of a bunch of parallel lines. For example, in $\mathbb{R}^3$, the equation $z = y^2$ defines a **parabolic cylinder**. (Since there is no restriction on x, that means x can have any value.) One way to think of this parabolic cylinder is a bunch of straight lines parallel to the x-axis (that all go through the parabola $z = y^2$ in the yz-plane). Another way to think of this parabolic cylinder is a bunch of 'copies' of a parabola that are all parallel to each other.

Quadric surfaces and traces

We've seen that in $\mathbb{R}^3$, a single equation of the form $ax + by + cz = d$ (where a, b, c, d are constants) will define a *plane*. Notice that a plane in $\mathbb{R}^3$ is an example of a two-dimensional surface that lives in a three-dimensional space. (Informally, one equation means putting one restriction on our variables, giving us a shape that has 'one dimension less' than the whole space.)

Similarly, we might have a slightly more complicated single equation containing x, y, and z. For instance, one or more of the variables might be squared. A single equation of this type will define a two-dimensional surface. If the equation is a second-degree polynomial in x, y, and z, then the surface is called a **quadric surface**.

Three dimensions is hard! A useful tool for visualizing a surface in $\mathbb{R}^3$ is **traces**, which are cross-sections parallel to one of the coordinate planes, or equivalently, cross-sections of the form $x = c$, $y = c$, or $z = c$. The traces that are in the coordinate planes themselves (i.e., by taking $c = 0$) are called the xy-trace, xz-trace, and yz-trace.

EXAMPLE 9: What does the surface defined by

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

look like? (This surface is called an 'ellipsoid'.)

EXAMPLE 10: What does the surface defined by

$$x^2 + y^2 = z^2$$

look like? (This surface is called a 'circular cone'.)

EXAMPLE 11: What does the surface defined by

$$z = \frac{x^2}{9} + \frac{y^2}{16}$$

look like? (This surface is called an 'elliptic paraboloid'.)

EXAMPLE 12: What does the surface defined by

$$z = \frac{x^2}{36} - \frac{y^2}{49}$$

look like? (This surface is called a 'hyperbolic paraboloid'.)