**Vector-valued functions and curves in space**

We can have an ordered triple $\langle x, y, z\rangle$ in three-dimensional space where each component $x, y, z$ is a function of a variable $t$.

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

It is often useful to think of $t$ as being time.

The three component functions make up a **vector-valued function** which traces out a **curve** in space.

We can write a vector-valued function in various ways:

$$\begin{aligned}\mathbf{r} = \mathbf{r}(t) &= \big\langle f(t), g(t), h(t)\big\rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\end{aligned}$$

**EXAMPLE 1:** Graph the vector-valued function.

$$\mathbf{r}(t) = \big\langle \cos t, \sin t, t\big\rangle$$

**Limits, continuity, and derivatives for vector-valued functions**

Limits are defined componentwise: if $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle$, then

$$\lim_{t\to c} \mathbf{r}(t) = \left\langle \lim_{t\to c} f(t),\ \lim_{t\to c} g(t),\ \lim_{t\to c} h(t) \right\rangle$$

A vector-valued function is continuous at a point if each of the component functions is continuous there.

The **derivative** of a vector-valued function is mathematically defined as

$$\mathbf{r}'(t) = \lim_{\delta\to 0} \frac{\mathbf{r}(t+\delta) - \mathbf{r}(t)}{\delta}$$

Notice that we are subtracting two *vectors* and dividing/multiplying by a *scalar*.

We can calculate the derivative componentwise:

If $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle$ then $\mathbf{r}'(t) = \big\langle f'(t), g'(t), h'(t) \big\rangle$.

Why? The key is that we have

$$\mathbf{r}(t+\delta) - \mathbf{r}(t) = \big\langle f(t+\delta) - f(t),\ g(t+\delta) - g(t),\ h(t+\delta) - h(t) \big\rangle.$$

So in Example 1, where we have

$$\mathbf{r}(t) = \big\langle \cos t, \sin t, t \big\rangle$$

we can differentiate each component to get

$$\mathbf{r}'(t) = \big\langle -\sin t, \cos t, 1 \big\rangle$$

What does this mean physically?

The vector $\mathbf{r}'(t)$ will be **tangent** to the curve.

**Physical interpretation of vector-valued functions**

$$\text{Position vector} = \mathbf{r}(t)$$

$$\text{Velocity vector} = \mathbf{v}(t) = \mathbf{r}'(t)$$

$$\text{Speed (scalar)} = |\mathbf{v}(t)| = |\mathbf{r}'(t)|$$

$$\text{Direction of motion} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{T}(t) \qquad \textbf{(unit tangent vector)}$$

$$\text{Acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

**EXAMPLE 2:** Find the velocity, speed, and acceleration of a particle whose position in three-dimensional space is given by

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

**EXAMPLE 3:** For the given curve, find an expression for the unit tangent vector at a general point.

$$\mathbf{r}(t) = \left\langle t\cos t,\ t\sin t,\ \frac{2\sqrt{2}}{3}t^{3/2} \right\rangle$$

**Derivative rules for vector-valued functions**

There are lots of these rules. Some examples are

$$\frac{d}{dt}\Big(\mathbf{u}(t)\cdot\mathbf{v}(t)\Big) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

$$\frac{d}{dt}\Big(\mathbf{u}(t)\times\mathbf{v}(t)\Big) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

Remember that for cross product, the order matters!

Why are those rules true? Suppose $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

We then have

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

$$\mathbf{u}\times\mathbf{v} = \Big\langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\Big\rangle$$

to which we can apply the product rule for *scalar* functions on the right.

**Integrals of vector-valued functions**

An antiderivative of a vector-valued function can be found componentwise.

When we find a general antiderivative, we add an unknown constant.

When finding an antiderivative of a *vector-valued* function, the constant is a constant *vector*.

We could write the antiderivative of a vector-valued function as

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

or in component form

$$\int \big\langle f(t), g(t), h(t) \big\rangle\,dt = \big\langle F(t), G(t), H(t) \big\rangle + \big\langle C_1, C_2, C_3 \big\rangle$$

where $F, G, H$ are antiderivatives of $f, g, h$ respectively.

**EXAMPLE 4:** Find the general antiderivative of the function.

$$\mathbf{r}(t) = \left\langle 6 - 6t,\ 3\sqrt{t},\ \frac{4}{t^2} \right\rangle$$

The *definite* integral of a vector-valued function can be found componentwise:

If $\mathbf{r}(t) = \big\langle f(t), g(t), h(t) \big\rangle$ then

$$\int_a^b \mathbf{r}(t)dt = \left( \int_a^b f(t)dt \right)\mathbf{i} + \left( \int_a^b g(t)dt \right)\mathbf{j} + \left( \int_a^b h(t)dt \right)\mathbf{k}.$$

**EXAMPLE 5:** Evaluate the definite integral.

$$\int_0^{\pi/3} \Big\langle \sec t \tan t,\ \tan t,\ 2\sin t \cos t \Big\rangle dt$$

**More physical applications of vector-valued functions**

**EXAMPLE 6:** A golf ball has an initial position of

$$\mathbf{r}_0 = \langle 0, 0, 0 \rangle$$

and an initial velocity of

$$\mathbf{v}_0 = \langle 70, 40, 80 \rangle$$

where distances are in feet and time is in seconds. If the acceleration due to gravity is

$$\mathbf{a}(t) = \mathbf{a} = \langle 0, 0, -32 \rangle$$

then express the future position of the golf ball as a function of time.

**Motion with constant $|\mathbf{r}|$**

Suppose a moving particle has position function $\mathbf{r} = \mathbf{r}(t)$, and suppose $|\mathbf{r}|$ is constant. This happens if the particle is constrained to lie on a circle (in $\mathbb{R}^2$) or sphere (in $\mathbb{R}^3$) with its center at the origin. Then $\mathbf{r} \cdot \mathbf{v} = 0$, which means the velocity vector is always perpendicular to the position vector.

Why? If $\mathbf{r}(t)$ has constant magnitude $c$, then $\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$ for all $t$. Then

$$\begin{aligned}
\frac{d}{dt}\Big(\mathbf{r}(t) \cdot \mathbf{r}(t)\Big) &= \frac{d}{dt}(c^2) = 0 \\
\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\
2\mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\
2\mathbf{r}(t) \cdot \mathbf{v}(t) &= 0 \\
\mathbf{r}(t) \cdot \mathbf{v}(t) &= 0
\end{aligned}$$

**Length of curves in three dimensions**

The length of a curve can be approximated by a large number of short diagonal line segments.

In two dimensions, a small piece of arc length is $ds = \sqrt{(dx)^2 + (dy)^2}$

In three dimensions, we have $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

The quantity $ds$ is sometimes called the (scalar) **element of arc length**.

So the arc length formula in three dimensions is

$$\text{Length of curve} = \int_{t=a}^{t=b} \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

We can write this more compactly as $\int_a^b |\mathbf{r}'(t)|\, dt$ or as $\int_a^b |\mathbf{v}(t)|\, dt$. This has a physical interpretation: The total length traveled is the integral of speed with respect to time.

**EXAMPLE 7:** Find the length of the portion of the curve

$$\mathbf{r}(t) = \left\langle 6\sin 2t,\ 6\cos 2t,\ 5t \right\rangle$$

between $t = 0$ and $t = \pi$.

**Arc length function**

We can measure arc length from a starting point to a general point.

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t |\mathbf{v}(u)|\,du$$

In some cases, this may be a nice function. We can parameterize a curve in terms of its arc length (although this might be difficult or impossible to do explicitly).

By the Fundamental Theorem of Calculus, the above formula implies

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = |\mathbf{v}(t)|$$

which has a physical interpretation: Derivative of length traveled with respect to time is equal to speed.

**EXAMPLE 8:** Find the arc length function for the curve defined by

$$\mathbf{r}(t) = \left\langle 6t^3,\ -2t^3,\ -3t^3 \right\rangle$$

If possible, reparameterize the curve in terms of its arc length.

If our curve is parameterized in a reasonable way, then the arc length function

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du$$

will be an increasing function of $t$, which means it is invertible, so in principle we can write $t$ as a function of $s$ (although in many cases, doing this explicitly will be difficult or impossible).

Parameterizing a curve with respect to arc length is equivalent to traveling along the curve at constant unit speed.