If a curve is parameterized in a reasonable way, then the arc length function

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

will be an increasing function of t, which means it is invertible, so in principle we can write t as a function of s (although in many cases, doing this explicitly will be difficult or impossible).

Parameterizing a curve with respect to arc length is equivalent to traveling along the curve at constant unit speed.

The unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ can be regarded as a function of the arc length s.

Curvature is defined as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

(the magnitude of the rate of change of the unit tangent vector as we travel along the curve at constant unit speed).

Curvature can be equivalently expressed as

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\,\frac{dt}{ds}\right| \\ &= \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right| \\ &= \frac{1}{|\mathbf{r}'(t)|}\left|\frac{d\mathbf{T}}{dt}\right| \qquad \text{or more briefly } \frac{|\mathbf{T}'|}{|\mathbf{r}'|}\end{aligned}$$

EXAMPLE 1: Find the curvature of the curve given below.

$$\mathbf{r}(t) = \langle 7 + 2t,\ -1 + 3t,\ 4 + 5t \rangle$$

EXAMPLE 2: Find the curvature of the curve given below.

$$\mathbf{r}(t) = \left\langle 5\cos t,\ 5\sin t,\ 0 \right\rangle$$

EXAMPLE 3: Find the curvature of the curve given below.

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, t \right\rangle$$

For some relatively simple curves, it is time-consuming to compute curvature using the formula $|\mathbf{T}'| / |\mathbf{r}'|$. (The tricky part is usually $\mathbf{T}'$.)

$$\mathbf{r}(t) = \langle t, t^2, 0\rangle \qquad \text{(parabola)}$$
$$\mathbf{r}(t) = \langle 3\cos t, 2\sin t, 0\rangle \qquad \text{(ellipse)}$$

Hence it is useful to have an alternative formula for curvature.

How do we get an alternative formula for curvature?

Recall that we often write $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{v}' = \mathbf{r}''$. Then $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$, and we have

$$\mathbf{v} = |\mathbf{v}|\,\mathbf{T}$$
$$\Longrightarrow \mathbf{a} = \mathbf{v}' = |\mathbf{v}|'\,\mathbf{T} + |\mathbf{v}|\,\mathbf{T}' \tag{1}$$

where we have used the version of the product rule for a scalar times a vector.

Since $\mathbf{T}$ is a *unit* vector, it has constant magnitude, implying that $\mathbf{T}'$ is orthogonal to $\mathbf{T}$. So equation (1) above can be written as

$$\mathbf{a} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1 = |\mathbf{v}|'\,\mathbf{T}$ is a vector parallel to $\mathbf{T}$ (and $\mathbf{v}$), and $\mathbf{u}_2 = |\mathbf{v}|\,\mathbf{T}'$ is a vector orthogonal to $\mathbf{T}$ (and $\mathbf{v}$). We can then form the cross product of $\mathbf{v}$ with $\mathbf{a}$:

$$\mathbf{v} \times \mathbf{a} = (\mathbf{v} \times \mathbf{u}_1) + (\mathbf{v} \times \mathbf{u}_2).$$

Since $\mathbf{v}$ and $\mathbf{u}_1$ are parallel, we have $\mathbf{v} \times \mathbf{u}_1 = 0$, giving us

$$\mathbf{v} \times \mathbf{a} = \mathbf{v} \times \mathbf{u}_2.$$

Since $\mathbf{v}$ and $\mathbf{u}_2$ are orthogonal, the angle θ between them is $\pi/2$, so we have

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v} \times \mathbf{u_2}| = |\mathbf{v}|\,|\mathbf{u}_2| \sin\theta = |\mathbf{v}|\,|\mathbf{u}_2| = |\mathbf{v}|^2\,|\mathbf{T}'|\,.$$

This implies

$$\frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{T}'|}{|\mathbf{v}|}$$

and the quantity on the left is our alternative formula for curvature.

Principal unit normal vector

As before, suppose $\mathbf{r}(t)$ is a vector-valued function that describes a curve. Then $\mathbf{T}(t) = \mathbf{r}'(t)/\left|\mathbf{r}'(t)\right|$ is the unit tangent vector to the curve.

Since $\mathbf{T}(t)$ is a unit vector, this means $\mathbf{T}(t)$ has constant magnitude, which implies that $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$, and hence also orthogonal to the curve.

Therefore the vector $\mathbf{T}'(t)/\left|\mathbf{T}'(t)\right|$ is a *unit* vector that is orthogonal to the curve. We write

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|}$$

and refer to this as the **principal unit normal vector** to the curve.