**Functions of several variables**

In Chapter 14, we dealt with vector-valued functions with one input variable.

Now, in Chapter 15, we deal with functions with several inputs.

Examples:

$$f(x,y) = x^2 + y^2 \qquad \text{scalar-valued function of two variables}$$

$$f(x,y) = \ln(1 - x^2 - y^2) \qquad \text{scalar-valued function of two variables}$$

$$f(x,y,z) = \frac{xyz}{x+y+z} \qquad \text{scalar-valued function of three variables}$$

What are the *domains* of each of these functions?

How can we visualize a function with two or three inputs?

One way to visualize a function of two variables $f(x, y)$ is to draw **level curves**. Those are curves of the form $f(x, y) = C$.

For example, let's do this for the function $f(x, y) = x^2 + y^2$.

(This could represent temperature as a function of location in two-dimensional space.)

Another way to visualize a function of two variables is to think of $f$ as height.

We could visualize $f(x, y) = x^2+y^2$ by drawing the graph of $z = x^2+y^2$ in three-dimensional space.

(In this case, the graph is a surface known as a 'paraboloid'. Compare with our discussion in Section 13.6.)

What about visualizing a function of *three* variables?

For example, $f(x, y, z) = x^2 + y^2 + z^2$

This might represent temperature as a function of location in three-dimensional space.

Equations of the form $f(x, y, z) = C$ determine *surfaces* in three-dimensional space, which we call **level surfaces**.

**Limits and continuity in higher dimensions**

For functions with two or three (or more) inputs, limits are harder.

In previous courses, you've seen various limit problems for functions of *one* variable.

$$\lim_{x\to 5} \frac{x-5}{x^2-25} = \frac{1}{10}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{|x|}{x} = \text{does not exist}$$

Limits of functions of *two* (or more) variables can be trickier. Example:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

If $f(x,y) = \dfrac{xy}{x^2+y^2}$, then $f(0,0)$ is undefined.

But what happens if $(x,y)$ *approaches* $(0,0)$?

In two (or more) dimensions, there are *infinitely* many ways that we could approach a point!

There are many many ways that $(x,y)$ could approach $(0,0)$.

$(x,y)$ *could* approach $(0,0)$ along the $x$-axis (so $y=0$ and $x=x$)

$(x,y)$ *could* approach $(0,0)$ along the $y$-axis (so $x=0$ and $y=y$)

$(x,y)$ *could* approach $(0,0)$ along various other lines or curves!

It turns out that $f(x,y) = \dfrac{xy}{x^2+y^2}$ approaches *different* values when $(x,y)$ approaches $(0,0)$ in different ways.

Therefore $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

This type of thing is not always obvious from just looking at the formula!

More limit problems to think about:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = ?$$

$$\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2) = ?$$

A tricky limit problem:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = ?$$

If the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

exists and is equal to $f(a,b)$, then we say the function $f(x,y)$ is continuous at the point $(a,b)$.

It is not always easy to prove from first principles that a function is continuous, but we do know that compositions of continuous functions are continuous.

For example, each of the functions

$$f(x,y) = e^{x-y}$$
$$f(x,y) = \cos \frac{xy}{x^2+1}$$
$$f(x,y) = \ln(1+x^2y^2)$$

is defined for all $(x,y)$, and is guaranteed to be continuous since it's 'built' out of continuous functions.

However, if a function is undefined at a point, it's not always obvious what happens near that point.