Partial derivatives

Given a function of several variables

$$f(x,y) \qquad \text{or} \qquad f(x,y,z)$$

we can take the derivative with respect to *one* of the inputs.

That means letting just *that* input change and the others remain constant.

For example,

$$\frac{d}{dy}\Big(f(x_0,y,z_0)\Big) = \lim_{\Delta y\to 0}\frac{f(x_0,y+\Delta y,z_0)-f(x_0,y,z_0)}{\Delta y}$$

and similarly for the other variables.

This represents rate of change of f *if* we move parallel to a coordinate axis.

EXAMPLE 1: If $f(x,y) = 9x^2y^4 - 2x^5y^3$, find $\dfrac{df}{dx}$ and $\dfrac{df}{dy}$.

Notation for partial derivatives

We often use curly d's:

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

We also sometimes use subscripts:

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y} \qquad f_z = \frac{\partial f}{\partial z}$$

EXAMPLE 2: If $g(x, y) = (x^2 + 5x - 2y)^8$, find $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$.

Higher-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Fact: If $f(x, y)$ *and* its partial derivatives are all continuous at a point, then the 'mixed' partial derivatives f_{xy} and f_{yx} will be equal there.

EXAMPLE 3: If $f(x, y) = x^2y^3 + x^4y^5$, find all second-order partial derivatives of f.

If f is a function of two (or more) variables, what do we mean when we say f is 'differentiable'?

For functions of *one* variable, it means there is a well-defined tangent *line*.

For functions of *two* variables, it means there is a well-defined tangent *plane*.

Compare and contrast the following two examples.

$f(x, y) = x^2 + y^2$ (paraboloid)

$f(x, y) = \sqrt{x^2 + y^2}$ (top half of cone)

The first is differentiable at *all* points, and the second is differentiable everywhere *except* $(0, 0)$.

The chain rule

Example: Suppose w is a function of three variables x, y, z and then each of x, y, z is a function of t.

Then $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$.

An excellent way to visualize this is to draw a diagram showing the 'dependencies' of the variables.

w depends on x, y, z, and each of x, y, z depends on t

A change in t can cause changes in x, y, z, each of which can cause changes in w.

Another example: Suppose w is a function of x, y, z, and each of x, y, z is a function of *two* variables u and v.

In that example, we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Idea: We must consider all possible 'routes' through which a change in u or v could cause a change in w.

EXAMPLE 4: If we are given

$$z = 8x^2y - 2x + 3y$$
$$x = uv$$
$$y = u - v$$

then find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Directional derivatives and gradients

Suppose $f(x, y)$ is a function of two variables.

Recall the partial derivatives $f_x = \dfrac{\partial f}{\partial x}$ and $f_y = \dfrac{\partial f}{\partial y}$.

Informally: f_x = rate of change of f if we move slightly in the x direction.

Similarly for f_y.

Directional derivative of f at (x_0, y_0) in the direction $\mathbf{u}$ (where $\mathbf{u}$ is a unit vector)

$$D_{\mathbf{u}} f(x_0, y_0)$$

Idea: Rate of change of f with respect to s if (x, y) moves a small distance s from (x_0, y_0) in the direction of $\mathbf{u}$.

Mathematical definition

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{ds}\Big[f\Big((x_0, y_0) + s\mathbf{u}\Big)\Big]$$

Similar definition for functions of three (or more) variables.

How do we *compute* directional derivatives?

In two dimensions: Say $\mathbf{u} = (u_1, u_2)$.

Then $D_\mathbf{u} f = f_x u_1 + f_y u_2$.

In three dimensions: Say $\mathbf{u} = (u_1, u_2, u_3)$.

Then $D_\mathbf{u} f = f_x u_1 + f_y u_2 + f_z u_3$.

Note: Those can be thought of as *dot products*.

$$f_x u_1 + f_y u_2 = (f_x, f_y) \cdot (u_1, u_2)$$
$$f_x u_1 + f_y u_2 + f_z u_3 = (f_x, f_y, f_z) \cdot (u_1, u_2, u_3)$$

The vector (f_x, f_y) or (f_x, f_y, f_z) is called the **gradient** of f.

The gradient of f is denoted ∇f.

$$\nabla f = (f_x, f_y) \qquad \text{or} \qquad \nabla f = (f_x, f_y, f_z)$$

The symbol ∇ is pronounced ‘gradient’ or ‘grad’ or ‘del’ or ‘nabla’.

The directional derivative can be expressed more briefly as

$$D_\mathbf{u} f = \nabla f \cdot \mathbf{u}$$

EXAMPLE 5: Find the directional derivative of $f(x, y) = 4x^3y^2$ at $(x_0, y_0) = (2, 1)$ in the direction of $\mathbf{a} = (4, -3)$.

Relationship between gradients and level curves / level surfaces

For example, consider $f(x, y) = x^2 + y^2$.

We previously saw that the level curves are circles centered at the origin.

Suppose you're standing at the point $(x_0, y_0) = (2, 3)$.

How would f change if you moved in various directions?

Remember, directional derivative is a dot product.

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

Suppose you're standing at (x_0, y_0), trying to decide which direction to go.

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \ |\mathbf{u}| \ \cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$.

Directional derivative is $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\ |\mathbf{u}|\ \cos\theta$

Directional derivative is *positive* if θ is between 0 and $\pi/2$.

Directional derivative is *zero* if θ is equal to $\pi/2$.

Directional derivative is *negative* if θ is between $\pi/2$ and π.

Directional derivative is *largest* if $\mathbf{u}$ has exactly the same direction as the gradient.

Directional derivative is *smallest* if $\mathbf{u}$ has exactly opposite direction to gradient.

Gradients are *perpendicular* to level curves or level surfaces.

The gradient points in the direction of steepest increase of f.