The tangent plane to a surface

Suppose we have a function of three variables. For example,

$$f(x,y,z) = x^2 + 4y^2 + z^2$$

Then surfaces of the form $f(x,y,z) = C$ are the level surfaces.
For example, $x^2 + 4y^2 + z^2 = 18$.

Note that $(1,2,1)$ is an example of a point on this surface.

How do we find the **tangent plane** to the surface $f(x,y,z) = C$ at the point $P_0 = (x_0, y_0, z_0)$?

Key: The gradient is always perpendicular to the level surface.

So the gradient is also perpendicular to the tangent plane.

To get an equation of a plane, we need a normal vector to the plane.

The 'normal line' to a surface at a point is the line through that point perpendicular to the surface.

If an equation of the surface is $f(x,y,z) = C$, then the normal line would be in the direction of the gradient at that point.

EXAMPLE 1: Find the tangent plane and normal line to the given surface at the given point.

$$x^2 + 2xy - y^2 + z^2 = 7, \qquad P_0 = (1, -1, 3)$$

A surface can also be given in the form $z = f(x, y)$.

EXAMPLE 2: Find the tangent plane and normal line to the surface $z = e^{-(x^2+y^2)}$ at the point $(0, 0, 1)$.

One way to do this is to rewrite in the form $f(x, y) - z = 0$.

Another way to think about tangent planes when $z = f(x, y)$

$$(\text{change in } f) = (\text{change in } f \text{ caused by } x) + (\text{change in } f \text{ caused by } y)$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This lets us approximate a nonlinear function with a linear one.

(Note that an equation of the form $z = Ax + By + C$ defines a plane.)

EXAMPLE 3: Find an equation of the tangent plane to the surface $z = \sqrt{y - x}$ at the point $(1, 2, 1)$.

Then use this to estimate the change in z if x changes from 1 to 0.998 and y changes from 2 to 2.006.

Extreme values and saddle points

Suppose f is a function of two variables x and y.

$f(a, b)$ is a **local maximum** if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b).

$f(a, b)$ is a **local minimum** if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b).

How do we **determine** if we have a max, min, or neither?

A **critical point** for a function $f(x, y)$ is a point where the partial derivatives f_x and f_y are both 0, or where one or both of the partial derivatives fails to exist.

Critical points are **candidates** for extrema. They could be a max, a min, or neither.

EXAMPLE 4: Find all critical points for each of the following functions.

$f(x, y) = x^2 + y^2$

$g(x, y) = -x^2 - y^2$

$h(x, y) = x^2 - y^2$

We say f has a **saddle point** at (a, b) if every neighborhood of (a, b) contains some (x, y) with $f(x, y) > f(a, b)$ and some (x, y) with $f(x, y) < f(a, b)$.

In other words, a saddle point is neither a max nor a min. If you're standing at a saddle point, then some nearby points are above you and some nearby points are below you.

EXAMPLE 5: Find all critical points of the function.

$$f(x, y) = y^2 + xy + 3y + 2x + 3$$

Second derivative test for local extrema

Suppose f and its derivatives are continuous.

Calculate $D = f_{xx}f_{yy} - f_{xy}^2$, which is the same as the determinant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

If D is positive, then 'yes, it's an extremum', then **one more step**

If D is negative, then 'not an extremum', i.e. 'saddle point'

If D is zero, then 'inconclusive', WE DON'T KNOW YET

The 'one more step' is: Look just at f_{xx}.

$f_{xx} > 0 \implies$ min, $f_{xx} < 0 \implies$ max

EXAMPLE 6: Apply the second derivative test to each of the following.

$f(x, y) = x^2 + y^2$

$g(x, y) = -x^2 - y^2$

$h(x, y) = x^2 - y^2$

EXAMPLE 7: Find and classify all critical points of the function.

$$f(x, y) = 6x + 6y - x^2 - y^2$$

EXAMPLE 8: Find and classify all critical points of the function.

$$f(x, y) = 3xy - x^3 - y^3$$

Absolute maxima and minima on **bounded** regions

Summary: You need to check

All **critical** points in the **interior** of the region

All points on the **boundary** of the region.

EXAMPLE 9: Find the absolute maximum and absolute minimum of $f(x, y) = xy - x - 2y$ in the triangular region bounded by

$$x = 0, \qquad y = 0, \qquad x + y = 4.$$