Multiple integrals

We already know about single integrals $\int_a^b f(x)\,dx$.

Here, $[a, b]$ is an interval and dx is a small change in x.

We can think of $f(x) \cdot dx = (\text{height}) \cdot (\text{width})$ as a small area.

Now suppose $f(x, y)$ is a function of *two* variables.

The inputs (x, y) are points in the xy-plane or 'floor'.

We consider double integrals over **regions** in the plane.

$$\iint_R f(x, y)\,dA$$

$R =$ name of region, $dA =$ 'element of area'

We could think of $f(x, y)$ as the 'height' above the point (x, y).

Then $f(x, y) \cdot dA = (\text{height}) \cdot (\text{area}) = \text{small piece of volume}$.

We could also think of $f(x, y)$ as temperature, density, etc.

How do we *evaluate* double integrals?

Simplest is if the region is a rectangle, say $a \leq x \leq b$ and $c \leq y \leq d$.

Then the double integral $\displaystyle\iint_R f(x,y)\,dA$ is equal to an *iterated* integral:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

We can evaluate an iterated integral from the 'inside out'.

When integrating with respect to one variable, we temporarily regard the other as constant.

EXAMPLE 1: Evaluate the integral $\displaystyle\int_0^3 \int_{-3}^0 (x^2y - 2xy)\,dy\,dx$

EXAMPLE 2: Evaluate the integral.

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\,dx\,dy$$

Double integrals over more general regions

Type 1 region: Bounds on x are constants, bounds on y are functions of x

Type 2 region: Bounds on y are constants, bounds on x are functions of y

For a Type 1 region, the bounds have the following form.

$$a \le x \le b$$
$$g_1(x) \le y \le g_2(x)$$

For a Type 2 region, the bounds have the following form.

$$c \le y \le d$$
$$h_1(y) \le x \le h_2(y)$$

In both cases, the bounds on the *outermost* variable must be constants.

EXAMPLE 3: Evaluate the integral

$$\iint_R (x+y)\,dA$$

where R is the region bounded by $y = x^2$ and $y = \sqrt{x}$.

Reversing the order of integration

Some regions can be described as Type 1 or Type 2.

EXAMPLE 4: Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

$$\int_0^1 \int_2^{4-2x} 5\, dy\, dx$$

EXAMPLE 5: Sketch the region of integration, reverse the order of integration, and evaluate the integral.

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$$

Area and average value

The **area** of a closed bounded region R is

$$\iint_R 1\,dA = \iint_R dA$$

The **average value** of a function f over a region R is

$$\frac{1}{(\text{area of } R)} \cdot \iint_R f(x,y)\,dA$$

EXAMPLE 6: Sketch the region bounded by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$. Then express the region's area as an iterated double integral and evaluate the integral.

EXAMPLE 7: Find the average value of $f(x, y) = \sin(x + y)$ over

(i) the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$

(ii) the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi/2$

EXAMPLE 8: Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1, 0 \leq y \leq 1$, or the average value of f over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant?

Calculate them to find out.