Double integrals in polar coordinates

Some regions in the plane are easier to describe in polar coordinates than in rectangular coordinates.

Some examples:

$0 \leq r \leq 5$ and $0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$

$2 \leq r \leq 3$ and $0 \leq \theta \leq \pi$

In those three examples, the bounds on r and θ were all constants.

We can also have equations where r is a function of θ.

One example: The curve $r = 1 + \cos\theta$ ('cardioid')

We could have a region in the plane bounded by **two** functions of the form $r = f(\theta)$ and $r = g(\theta)$. (Maybe one of the functions is 0, but maybe not.)

We sometimes have a function f that we need to integrate over a region R in the xy plane.

$$\iint_R f(x,y)\,dA$$

When using rectangular coordinates, we have $dA = dx\,dy = dy\,dx$.

What's the correct element of area in **polar** coordinates?

IMPORTANT FACT: In polar coordinates, the element of area is

$$dA = r\,dr\,d\theta$$

Loosely speaking, it looks like there's an 'extra' r, but it's there for a very good reason.

EXAMPLE 1: Evaluate the integral.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

EXAMPLE 2: Evaluate the integral.

$$\int_1^{\sqrt{3}} \int_1^x dy\, dx$$

EXAMPLE 3: Evaluate the integral.

$$\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$$