Triple integrals in rectangular coordinates

If G is a region in *three*-dimensional space, we write

$$\iiint_G f(x, y, z)\, dV$$

to mean the integral of $f(x, y, z)$ over the region G.

Here, f is a scalar-valued function (e.g. density or temperature as a function of location in space)

One common situation: G is bounded by a 'bottom surface' $z = g_1(x, y)$ and a 'top surface' $z = g_2(x, y)$, and lies above* a two-dimensional region R in the xy plane.

If G is this type of three-dimensional region, then the bounds on z depend on x and y.

In that case, z must go *inside* x and y in the order of integration.

EXAMPLE. Suppose we want to integrate some function over the interior of the sphere $x^2 + y^2 + z^2 \leq 1$.

Boundary $x^2 + y^2 + z^2 = 1 \implies z^2 = 1 - x^2 - y^2$

'Top surface' $z = \sqrt{1 - x^2 - y^2}$

'Bottom surface' $z = -\sqrt{1 - x^2 - y^2}$

So in that case we would have

$$\iiint_G f(x,y,z)\,dV = \iint_R \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz \right) dA$$

where R is the unit disk in the xy-plane.

For example, if we want to find the **volume** of G, we would integrate $f(x,y,z) = 1$.

EXAMPLE 1: Evaluate the integral.

$$\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$$

EXAMPLE 2: Evaluate the integral.

$$\int_0^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\,dx\,dy$$

Cylindrical coordinates

Cylindrical coordinates are 'polar coordinates with z'.

Relationship between rectangular and cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

How to integrate in cylindrical coordinates?

IMPORTANT FACT: In cylindrical coordinates, the element of volume is

$$dV = r\,dz\,dr\,d\theta$$

Just as with polar coordinates in two dimensions, there is a seemingly 'extra' r. However, it is there for a very good reason (a geometric reason).

Spherical coordinates

Spherical coordinates (ρ, ϕ, θ) consist of a radius and two angles.

$$\begin{aligned}\rho &= \text{distance from the origin}\\ \phi &= \text{angle measured from the positive } z\text{-axis}\\ \theta &= \text{angle in the } xy \text{ plane, like before}\end{aligned}$$

Relationship between rectangular and spherical coordinates

$$\begin{aligned}x &= \rho \sin\phi \cos\theta\\ y &= \rho \sin\phi \sin\theta\\ z &= \rho \cos\phi\end{aligned}$$

Why is this? One possible answer: Take cylindrical coordinates, and replace r with $\rho \sin\phi$ and replace z with $\rho \cos\phi$.

Also notice: We go *once* around the whole sphere if $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

How to integrate in spherical coordinates?

IMPORTANT FACT: In spherical coordinates, the element of volume is

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

So now, there's an 'extra' $\rho^2 \sin\phi$. Why is that? There is both a geometric and an algebraic reason.

EXAMPLE 3: Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Find the volume of D using cylindrical coordinates.

EXAMPLE 4: Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the volume of D using spherical coordinates.

Where does the 'extra' r come from in polar and cylindrical coordinates, and the 'extra' $\rho^2 \sin\phi$ in spherical coordinates?

Suppose we do *any* change of variables in two dimensions. Suppose the 'old' variables x and y are rewritten using two 'new' variables u and v.

The **Jacobian** of the transformation is the following determinant.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

In three dimensions, suppose the 'old' variables x, y, z are rewritten using 'new' variables u, v, w. The Jacobian in that case is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Let's calculate the Jacobian for each of the following transformations.

Rectangular to polar (two dimensions)

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Rectangular to cylindrical (three dimensions)

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Rectangular to spherical (three dimensions)

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$