Vector fields

A vector field in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is a function that assigns a vector to each point in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

In other words, a vector field is a vector-valued function.

We can think of a vector field as being like wind, or the flow of a fluid.

Other examples are forces due to gravity or magnetism.

EXAMPLE: $\mathbf{F}(x, y) = (-y, x)$ in $\mathbb{R}^2$.

What are some ways to generate a vector-valued function?

Remember **gradients**. If $f = f(x, y, z)$ is a scalar-valued function, then its gradient is defined as

$$\nabla f = (f_x, f_y, f_z)$$

which is a vector-valued function.

So, a vector field $\mathbf{F}$ *could* be the gradient of some scalar-valued function f, but it doesn't have to be. If $\mathbf{F}$ is the gradient of f, we call $\mathbf{F}$ a **gradient field** and we call f a **potential function** for $\mathbf{F}$.

Line integrals

Line integrals of **scalar**-valued functions

Suppose $f = f(x, y, z)$ is a scalar-valued function (in 3D space).

Suppose C is a curve in space, parametrized by $\big(g(t), h(t), k(t)\big)$.

Then we can define the **line integral**

$$\int_C f(x, y, z)\, ds$$

Informally, we're adding $f(x, y, z)\, ds$ where ds is a small change in the length.

Notice that ds is a **scalar**.

How do we **evaluate** a line integral?

Parametrize C as $\mathbf{r}(t) = (x, y, z) = \big(g(t), h(t), k(t)\big)$ with $a \le t \le b$.

Then $ds = |\mathbf{v}(t)|\ dt = |\mathbf{r}'(t)|\ dt$.

And then $\displaystyle\int_C f(x, y, z)\, ds = \int_{t=a}^{t=b} f\big(g(t), h(t), k(t)\big)\, |\mathbf{r}'(t)|\, dt$.

EXAMPLE: Evaluate the integral

$$\int_C (x - y + z - 2)\, ds$$

where C is the straight line segment

$$\begin{aligned} x &= t \\ y &= 1 - t \\ z &= 1 \end{aligned}$$

from $(0, 1, 1)$ to $(1, 0, 1)$.

Line integral of a **vector**-valued function

Say $\mathbf{F}$ is a vector field, and C is a curve parametrized by $\mathbf{r}(t)$ for $a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt$$

(Notice we are taking the dot product of two vectors in the integral.)

If $\mathbf{F}$ is a force field, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is the **work** done by the force $\mathbf{F}$ along the curve C.

If $\mathbf{F}$ is a velocity field, then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is called the **flow** along the curve.

(If the curve starts and ends at the same point, the flow is called the **circulation**.)

EXAMPLES:

Here are two different paths from $(0,0,0)$ to $(1,1,1)$.

C_1 = the straight-line path $\mathbf{r}(t) = (t,t,t)$ where $0 \le t \le 1$

C_2 = the curved path $\mathbf{r}(t) = (t,t^2,t^4)$ where $0 \le t \le 1$

Find the line integral of $\mathbf{F}$ along C_1 and along C_2, if:

- $\mathbf{F}(x,y,z) = (3y, 4x, 2z)$
- $\mathbf{F}(x,y,z) = (\sqrt{z}, -2x, \sqrt{y})$

Different ways to write a line integral of a vector field

If $\mathbf{F} = (f, g, h)$ and C is parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t))$, then the line integral of $\mathbf{F}$ along C can be written in the following ways:

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f\, dx + g\, dy + h\, dz$$

We can also write

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

To see this, recall that the unit tangent vector is $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and that we have $ds = |\mathbf{r}'(t)|\; dt$, so $\mathbf{T}\, ds = \mathbf{r}'(t)\, dt$.

Recall that the above integral is called the **flow** of $\mathbf{F}$ along C.

If C is a smooth simple closed curve in the plane, then we can also define the integral

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds$$

where $\mathbf{n}$ is the outward-pointing normal vector on C.

This integral is called the **flux** of $\mathbf{F}$ **across** C.

FACT: If $\mathbf{F} = (M, N)$ and the curve C is traversed counterclockwise, then

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C M\, dy - N\, dx$$

Why? This is explained in the textbook, but roughly summarized, it follows from $\mathbf{n} = \mathbf{T} \times \mathbf{k}$.

EXAMPLES:

Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \qquad \text{and} \qquad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

- The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$
- The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

Path independence and conservative fields

If $\mathbf{F}$ is a vector field and $\mathbf{F} = \nabla f$ for some scalar-valued function f, then f is called a **potential function** for $\mathbf{F}$.

If $\mathbf{F}$ has a potential function, then $\mathbf{F}$ is called **conservative**.

FACT: Suppose C is a smooth curve from point A to point B, and suppose f is a potential function for $\mathbf{F}$. We then have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

In other words, **if** the vector field $\mathbf{F}$ has a potential function, then the above integral is **independent of path**.

EXAMPLE: Which of the following vector fields are conservative? For those that are conservative, find a potential function.

$$\mathbf{F} = (yz, xz, xy)$$
$$\mathbf{F} = (y \sin z, x \sin z, xy \cos z)$$
$$\mathbf{F} = (y, x + z, -y)$$

EXAMPLE: Evaluate the integral by finding a potential function.

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) dy + \frac{1}{z} \, dz$$

A necessary and sufficient condition for $\mathbf{F} = (M, N, P)$ to be conservative:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \text{and} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$