Green's Theorem

An integral along a simple closed curve in a counterclockwise direction

$$\oint$$

Green's Theorem is a relationship between such an integral and a double integral. There are two forms of Green's Theorem.

Green's Theorem – Circulation Form

If C is a simple closed curve enclosing a region R, and $\mathbf{F}(x,y) = \langle f, g\rangle$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

The quantity $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}$ is called the **two-dimensional curl** of the vector field $\mathbf{F}$.

Roughly speaking, the net circulation around C is equal to the total of the circulation at all points inside C.

If a vector field $\mathbf{F} = \langle f, g\rangle$ satisfies $\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} = 0$ at all points, then $\mathbf{F}$ is called **irrotational**.

Fact: Irrotational vector fields are conservative.

Green's Theorem – Divergence Form

If C is a simple closed curve enclosing a region R, and $\mathbf{F}(x,y) = \langle f, g \rangle$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C f\, dy - g\, dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx\, dy$$

The quantity $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$ is called the **two-dimensional divergence** of the vector field $\mathbf{F}$.

Informally, $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$ is a measure of the 'expansiveness' or 'outward flux' of $\mathbf{F}$.

If a vector field $\mathbf{F} = \langle f, g \rangle$ satisfies $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0$ at all points, then $\mathbf{F}$ is called **source-free**.