The problem: Evaluate the limit.

$$\lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{n^n}-\frac{n^n}{(n-1)^{n-1}}\right]$$

We first explore some slightly imprecise reasoning that unfortunately leads to two different answers (although one of them does *turn out* to be correct).

$$\begin{aligned}
&\frac{(n+1)^{n+1}}{n^n} \;-\; \frac{n^n}{(n-1)^{n-1}}\\
&=\frac{(n+1)^n}{n^n}\cdot(n+1) \;-\; \frac{n^{n-1}}{(n-1)^{n-1}}\cdot n\\
&=\left(1+\frac{1}{n}\right)^n\cdot(n+1) \;-\; \left(1+\frac{1}{n-1}\right)^{n-1}\cdot n
\end{aligned}$$

which, when n is large, informally 'behaves like' $e\cdot(n+1)-e\cdot n=e$. However, note that we also have

$$\begin{aligned}
&\frac{(n+1)^{n+1}}{n^n} \;-\; \frac{n^n}{(n-1)^{n-1}}\\
&=\frac{(n+1)^n}{n^n}\cdot(n+1) \;-\; \frac{n^n}{(n-1)^n}\cdot(n-1)\\
&=\left(1+\frac{1}{n}\right)^n\cdot(n+1) \;-\; \left(\frac{n-1}{n}\right)^{-n}\cdot(n-1)\\
&=\left(1+\frac{1}{n}\right)^n\cdot(n+1) \;-\; \left(1-\frac{1}{n}\right)^{-n}\cdot(n-1)
\end{aligned}$$

Recalling that $\lim_{n\to\infty}(1-\frac{1}{n})^n=e^{-1}$, we see that when n is large, our expression informally 'behaves like' $e\cdot(n+1)-e\cdot(n-1)=2e$.

So which is it? The difficulty is that limits that informally 'behave like' a difference of two linear functions of n are actually of the indeterminate form $\infty-\infty$, and we have to be careful with informal reasoning about subtracting infinities. (A function that 'grows like' a linear function of n could potentially have an 'error term' that grows like, say, the square root of n.)

There is probably more than one way to determine the limit rigorously. Here is one.

We define

$$G(n) = \frac{(n+1)^{n+1}}{n^n}$$

so the problem is to evaluate the limit

$$\lim_{n\to\infty} \big[G(n) - G(n-1)\big].$$

Using the mean value theorem, we can write

$$G(n) - G(n-1) = \frac{G(n) - G(n-1)}{n - (n-1)} = G'(\xi)$$

for some ξ between $n-1$ and n. We now focus on bounding $G'(x)$ for a general positive x. We have

$$\ln G(x) = (x+1)\ln(x+1) - x\ln x$$

and differentiating both sides with respect to x gives

$$\begin{aligned} \frac{G'(x)}{G(x)} &= 1\ln(x+1) + \frac{x+1}{x+1} - 1\ln x - \frac{x}{x} \\ &= \ln\left(\frac{x+1}{x}\right) = \ln\left(1 + \frac{1}{x}\right) \end{aligned}$$

so we have

$$G'(x) = G(x)\ln\left(1+\frac{1}{x}\right) = \frac{(x+1)^{x+1}}{x^x}\ln\left(1+\frac{1}{x}\right). \tag{1}$$

We now take it as known that for small positive t, we have

$$\ln(1+t) = t + O(t^2)$$

so for large positive x, we have

$$\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} + O\Big(\frac{1}{x^2}\Big).$$

If we want to be more explicit than that, we can say

$$t - \frac{t^2}{2} \le \ln(1+t) \le t,$$

$$\frac{1}{x} - \frac{1}{2x^2} \le \ln\left(1+\frac{1}{x}\right) \le \frac{1}{x}.$$

Now, equation (1) can be rewritten

$$\begin{aligned}
G'(x) &= \frac{(x+1)^{x+1}}{x^x}\left[\frac{1}{x} + O\Big(\frac{1}{x^2}\Big)\right] \\
&= \Big(\frac{x+1}{x}\Big)^x (x+1)\left[\frac{1}{x} + O\Big(\frac{1}{x^2}\Big)\right] \\
&= \Big(\frac{x+1}{x}\Big)^x \left[\frac{x+1}{x} + (x+1)O\Big(\frac{1}{x^2}\Big)\right] \\
&= \Big(1+\frac{1}{x}\Big)^x \left[1 + \frac{1}{x} + O\Big(\frac{1}{x}\Big)\right]
\end{aligned}$$

which approaches e as x approaches infinity.

Therefore for every $\varepsilon > 0$, $G'(x)$ eventually stays within ε of e, so we can conclude that if n is sufficiently large, then $G'(\xi)$ is within ε of e.