Part 2

Limits

Question 2.1. Find the limit.

$$\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$$

First: If we TRY $t=-1$ we get $\dfrac{(-1)^2+3(-1)+2}{(-1)^2-(-1)-2} = \dfrac{1-3+2}{1+1-2} = \dfrac{0}{0}$

So we DON'T KNOW THE ANSWER YET.

Method 1: Algebra (factoring). $\displaystyle\lim_{t\to -1} \frac{(t+1)(t+2)}{(t+1)(t-2)} = \lim_{x\to -1} \frac{t+2}{t-2}$

$$= \frac{-1+2}{-1-2} = \frac{1}{-3} = \boxed{-\frac{1}{3}}$$

Method 2: L'Hopital's rule. $f(t) = t^2+3t+2 \Rightarrow f'(t) = 2t+3$

$g(t) = t^2-t-2 \Rightarrow g'(t) = 2t-1$

So limit $\displaystyle = \lim_{t\to -1} \frac{2t+3}{2t-1} = \frac{2(-1)+3}{2(-1)-1} = \frac{-2+3}{-2-1} = \frac{1}{-3} = \boxed{-\frac{1}{3}}$

Question 2.2. Find the limit.

$$\lim_{x \to -2} \frac{-2x - 4}{x^3 + 2x^2}$$

If we TRY $x=-2$ we get $\dfrac{-2(-2)-4}{(-2)^3+2(-2)^2} = \dfrac{4-4}{-8+2\cdot 4} = \dfrac{0}{0}$

SO WE DON'T KNOW YET.

Method 1: $\displaystyle \lim_{x\to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x\to -2} \frac{-2}{x^2} = \frac{-2}{(-2)^2}$

$= \dfrac{-2}{4} = \boxed{\dfrac{-1}{2}}$

Method 2: $\frac{0}{0}$ form $\Rightarrow$ can use L'Hopital

$f(x) = -2x-4 \Rightarrow f'(x) = -2$

$g(x) = x^3+2x^2 \Rightarrow g'(x) = 3x^2+4x$

So limit $= \displaystyle \lim_{x\to -2} \frac{-2}{3x^2+4x} = \frac{-2}{3(-2)^2+4(-2)}$

$= \dfrac{-2}{3\cdot 4 - 4\cdot 2} = \dfrac{-2}{12-8} = \dfrac{-2}{4} = \boxed{\dfrac{-1}{2}}$

Question 2.3. Find the limit.

$$\lim_{x\to 1} \frac{x^{-1}-1}{x-1}$$

If we TRY $x=1$ we get $\dfrac{1^{-1}-1}{1-1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ DON'T KNOW YET

Method 1 (algebra). $\displaystyle \lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} \cdot \frac{x}{x} = \lim_{x\to 1} \frac{\frac{x}{x}-x}{(x-1)x}$

$\displaystyle = \lim_{x\to 1} \frac{1-x}{(x-1)x} = \lim_{x\to 1} \frac{-1(x-1)}{(x-1)x} = \lim_{x\to 1} \frac{-1}{x}$

$\displaystyle = \frac{-1}{1} = \boxed{-1}$

Method 2. $\frac{0}{0} \Rightarrow$ can use L'Hopital.

$f(x) = x^{-1} - 1 \Rightarrow f'(x) = -1x^{-2}$

$g(x) = x - 1 \Rightarrow g'(x) = 1$

So limit $\displaystyle = \lim_{x\to 1} \frac{-1x^{-2}}{1} = \frac{-1\cdot 1^{-2}}{1} = \frac{-1\cdot 1}{1} = \boxed{-1}$

Question 2.4. Find the limit.

$$\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$$

If we TRY $x=0$ we get $\dfrac{\frac{1}{0-1}+\frac{1}{0+1}}{0} = \dfrac{-1+1}{0} = \dfrac{0}{0}$

Method 1 (algebra). $\displaystyle\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)}$

$$= \lim_{x\to 0} \frac{\frac{(x-1)(x+1)}{x-1}+\frac{(x-1)(x+1)}{x+1}}{x(x-1)(x+1)} = \lim_{x\to 0} \frac{(x+1)+(x-1)}{x(x-1)(x+1)}$$

$$= \lim_{x\to 0} \frac{2x}{x(x-1)(x+1)} = \lim_{x\to 0} \frac{2}{(x-1)(x+1)} = \frac{2}{(0-1)(0+1)}$$

$$= \frac{2}{-1\cdot 1} = \boxed{-2}$$

Method 2 (L'Hopital) $f(x) = (x-1)^{-1} + (x+1)^{-1}$

$$\Rightarrow f'(x) = -1(x-1)^{-2}\cdot 1 + (-1)(x+1)^{-2}\cdot 1$$

$g(x) = x \Rightarrow g'(x) = 1$

$$= \frac{-1}{(x-1)^2} + \frac{-1}{(x+1)^2}$$

New limit $= \displaystyle\lim_{x\to 0} \frac{\frac{-1}{(x-1)^2}+\frac{-1}{(x+1)^2}}{1} = \frac{\frac{-1}{(0-1)^2}+\frac{-1}{(0+1)^2}}{1} = \ldots = -2$

Question 2.5. Find the limit.

$$\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}$$

If we TRY $x=-1$ we get $\dfrac{\sqrt{(-1)^2+8}-3}{-1+1} = \dfrac{\sqrt{1+8}-3}{-1+1} = \dfrac{0}{0}$

Method 1. $\displaystyle\lim_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1}\cdot\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$

$$=\lim_{x\to-1}\frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to-1}\frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$=\lim_{x\to-1}\frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to-1}\frac{x-1}{\sqrt{x^2+8}+3}$$

$$=\frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{\sqrt{1+8}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{3+3}$$

$$=\frac{-2}{6} = \boxed{-\frac{1}{3}}$$

Method 2 (L'Hopital) would start with

$$f(x) = (x^2+8)^{1/2}-3$$

$$f'(x) = \frac{1}{2}(x^2+8)^{-1/2}\cdot 2x$$

This will work, because $g'(x)=1$.
Try to finish this for extra practice!

Question 2.6. Find the limit.

$$\lim_{x\to\infty}\sqrt{\frac{8x^2-3}{2x^2+x}}$$

Can we GUESS the answer? If x is "extreme", then $8x^2-3$ grows like $8x^2$, and $2x^2+x$ grows like $2x^2$

so $$\sqrt{\frac{8x^2-3}{2x^2+x}} \approx \sqrt{\frac{8x^2}{2x^2}} = \sqrt{\frac{8}{2}} = \sqrt{4} = \boxed{2}$$

More detailed solution:

$$\lim_{x\to\infty}\sqrt{\frac{8x^2-3}{2x^2+x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}} = \lim_{x\to\infty}\sqrt{\frac{8-\frac{3}{x^2}}{2-\frac{1}{x}}}$$

$$= \sqrt{\frac{8-0}{2-0}} = \sqrt{\frac{8}{2}} = \sqrt{4} = \boxed{2}$$

Question 2.7. Find the limit.

$$\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$$

If x is "extreme" then $\sqrt{x} = x^{1/2}$ approaches ∞

$x^{-1} = \frac{1}{x}$ approaches 0

Fastest growing function on top is $2x^{1/2}$

Fastest growing function in bottom is $3x$

Answer will be the same as $\lim_{x\to\infty} \frac{2\sqrt{x}}{3x} = \lim_{x\to\infty} \frac{2}{3\sqrt{x}} = \boxed{0}$

More detailed solution:

$$\lim_{x\to\infty} \frac{2x^{1/2}+\frac{1}{x}}{3x-7}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{2}{x^{1/2}}+\frac{1}{x^2}}{3-\frac{7}{x}}$$

$$= \frac{0+0}{3-0} = \frac{0}{3} = \boxed{0}$$

Question 2.8. Find the limit.

$$\lim_{x \to -\infty} \frac{x^{1/3} - 5x + 3}{2x + x^{2/3} - 4}$$

$x \to -\infty \Rightarrow x$ is "extreme"

In the top: $-5x^1$ is more extreme than $x^{1/3}$ or 3

In the bottom: $2x^1$ is more extreme than $x^{2/3}$ or -4

(x^1 is largest power of x in top or bottom)

So answer will be same as $\displaystyle \lim_{x \to -\infty} \frac{-5x}{2x} = \boxed{\frac{-5}{2}}$

More detailed solution:

$$\lim_{x \to -\infty} \frac{x^{1/3} - 5x + 3}{2x + x^{2/3} - 4} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x^{2/3}} - 5 + \frac{3}{x}}{2 + \frac{1}{x^{1/3}} - \frac{4}{x}} = \frac{0 - 5 + 0}{2 + 0 - 0}$$

$$= \boxed{\frac{-5}{2}}$$

Question 2.9. Find the limit.

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$$

$x \to -\infty \Rightarrow x$ is "extreme". If x is extreme, then top grows like $-3x^3$ and bottom grows like $\sqrt{x^6}$.

CAREFUL: Here, $\sqrt{x^6}$ is NOT the same as x^3.

Since $x \to -\infty$, that means x is negative. But $\sqrt{x^6}$ means the positive square root of x^6, which is $\underbrace{-\underbrace{x^3}_{\text{neg}}}_{\text{pos}}$.

Answer is $\displaystyle\lim_{x\to-\infty} \frac{-3x^3+4}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{-3x^3}{\sqrt{x^6}}$

$\displaystyle = \lim_{x\to-\infty} \frac{-3x^3}{-x^3} = \boxed{+3}$

(To "see" this more intuitively, could plug in $x = -1000$)

Question 2.10. Find the limit.

$$\lim_{x\to 0} \frac{\sin 5x}{x}$$

METHOD 1: Use the known fact that $\lim_{w\to 0} \frac{\sin w}{w} = 1$.

Given limit $= \lim_{x\to 0} \frac{\sin(5x)}{x} \cdot \frac{5}{5} = \lim_{x\to 0} \underbrace{\frac{\sin(5x)}{5x}}_{\text{has the form } \frac{\sin w}{w} \text{ where } w\to 0} \cdot 5$

$= 1\cdot 5 = \boxed{5}$

METHOD 2: If we try plugging in $x=0$, we get

$\frac{\sin(5\cdot 0)}{0} = \frac{\sin 0}{0} = \frac{0}{0}$. Don't know yet. Use L'Hopital.

$f(x) = \sin(5x) \Rightarrow f'(x) = \cos(5x)\cdot 5$ (CHAIN RULE!)

$g(x) = x \Rightarrow g'(x) = 1$

So limit $= \lim_{x\to 0} \frac{\cos(5x)\cdot 5}{1} = \frac{\cos(5\cdot 0)\cdot 5}{1}$

$= \frac{\cos(0)\cdot 5}{1} = \frac{1\cdot 5}{1} = \boxed{5}$

Question 2.11. Find the limit.

$$\lim_{x\to 0} \frac{8x^2}{\cos x - 1}$$

If we try plugging in $x=0$ we get $\frac{8\cdot 0^2}{\cos 0 - 1} = \frac{0}{1-1} = \frac{0}{0}$

L'Hopital. $f(x) = 8x^2 \Rightarrow f'(x) = 16x$

$g(x) = \cos x - 1 \Rightarrow g'(x) = -\sin x$

New limit is $\lim\limits_{x\to 0} \frac{16x}{-\sin x}$ Try $x=0$ and get $\frac{16\cdot 0}{-\sin 0} = \frac{0}{0}$

So we can do L'Hopital again.

$f(x) = 16x \Rightarrow f'(x) = 16$

$g(x) = -\sin x \Rightarrow g'(x) = -\cos x$

New limit is $\lim\limits_{x\to 0} \frac{16}{-\cos x} = \frac{16}{-\cos 0} = \frac{16}{-1}$

$= \boxed{-16}$

Question 2.12. Find the limit.

$$\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{1 + \cos 2\theta}$$

If we try $\theta = \frac{\pi}{2}$ we get $\dfrac{1 - \sin\frac{\pi}{2}}{1 + \cos(2\cdot\frac{\pi}{2})} = \dfrac{1-1}{1 + \underbrace{\cos\pi}_{-1}} = \dfrac{0}{0}$

L'Hopital: $f(\theta) = 1 - \sin\theta \Rightarrow f'(\theta) = -\cos\theta$

$g(\theta) = 1 + \cos(2\theta) \Rightarrow g'(\theta) = -\sin(2\theta)\cdot 2$

New limit is $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{-\cos\theta}{-2\sin 2\theta} = \lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{2\sin 2\theta}$

If we now try $\theta = \frac{\pi}{2}$ we get $\dfrac{\cos\frac{\pi}{2}}{2\underbrace{\sin(2\cdot\frac{\pi}{2})}_{\sin\pi\ =0}} = \dfrac{0}{2\cdot 0} = \dfrac{0}{0}$

L'Hopital again: $f(\theta) = \cos\theta \Rightarrow f'(\theta) = -\sin\theta$

$g(\theta) = 2\sin(2\theta) \Rightarrow g'(\theta) = 2\cos(2\theta)\cdot 2$

$= 4\cos 2\theta$

New limit: $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta}{4\cos 2\theta} = \frac{-\sin\frac{\pi}{2}}{4\cos(2\cdot\frac{\pi}{2})} = \frac{-1}{4\cdot(-1)} = \boxed{\frac{1}{4}}$

Question 2.13. Find the limit.

$$\lim_{t\to\infty} \frac{e^t + t^2}{e^t - t}$$

Guess answer: Fastest growing function in top is e^t

Fastest growing function in bottom is e^t

Answer is same as $\lim\limits_{t\to\infty} \frac{e^t}{e^t} = \boxed{1}$

Another method: Limit has form $\frac{\infty}{\infty}$, so use L'Hopital

$f(t) = e^t + t^2 \Rightarrow f'(t) = e^t + 2t$

$g(t) = e^t - t \Rightarrow g'(t) = e^t - 1$

New limit is $\lim\limits_{t\to\infty} \frac{e^t + 2t}{e^t - 1}$ which is $\frac{\infty}{\infty}$ again

If we do L'Hopital again we get $\lim\limits_{t\to\infty} \frac{e^t + 2}{e^t}$ ($\frac{\infty}{\infty}$ yet again)

If we do L'Hopital one more time

we get $\lim\limits_{t\to\infty} \frac{e^t}{e^t} = \boxed{1}$