Part 3

Derivatives

Question 3.1. Find the derivative of the function.

$$y = \frac{1}{18}(3x-2)^6 + \left(4 - \frac{1}{2x^2}\right)^{-1}$$

Rewrite: $$y = \frac{1}{18}\underbrace{(3x-2)^6}_{\text{composition}} + \underbrace{\left(4-\tfrac{1}{2}x^{-2}\right)^{-1}}_{\text{composition}}$$

$$\frac{dy}{dx} = \frac{1}{18}\cdot 6(3x-2)^5\cdot 3$$

$$+(-1)\left(4-\tfrac{1}{2}x^{-2}\right)^{-2}\cdot\tfrac{-1}{2}(-2)x^{-3}$$

$$= (3x-2)^5 - x^{-3}\left(4-\tfrac{1}{2}x^{-2}\right)^{-2}$$

Question 3.2. Find the derivative of the function.

$$y = (2x-5)^{-1}(x^2-5x)^6$$

$\uparrow$ Product $\quad y = u \cdot v$

$$\frac{dy}{dx} = \left((2x-5)^{-1}\right)'(x^2-5x)^6 + (2x-5)^{-1}\left((x^2-5x)^6\right)'$$

$$= -1(2x-5)^{-2} \cdot 2 \cdot (x^2-5x)^6 + (2x-5)^{-1} \cdot 6(x^2-5x)^5 \cdot (2x-5)$$

$$= -2(2x-5)^{-2}(x^2-5x)^6 + 6(x^2-5x)^5$$

$$= 2(x^2-5x)^5\left[-(2x-5)^{-2} + 3\right]$$

or $\quad 2(x^2-5x)^5 \cdot \left(3 - \frac{1}{(2x-5)^2}\right)$

Question 3.3. Find the derivative of the function.

$$y = (9x^2 - 6x + 2)e^{x^3}$$

$$\frac{dy}{dx} = (9x^2-6x+2)' e^{x^3} + (9x^2-6x+2)(e^{x^3})'$$

$$= (18x-6)e^{x^3} + (9x^2-6x+2)\cdot e^{x^3} \cdot 3x^2$$

$$= (18x-6)e^{x^3} + (27x^4-18x^3+6x^2)e^{x^3}$$

$$= (27x^4-18x^3+6x^2+18x-6)e^{x^3}$$

$$= 3(9x^4-6x^3+2x^2+6x-2)e^{x^3}$$

Question 3.4. Find the derivative of the function.

$$h(x) = x\tan(2\sqrt{x}) + 7$$

$$h'(x) = (x)'\tan(2\sqrt{x}) + x\cdot\left(\tan(2\sqrt{x})\right)'$$

$$= 1\cdot\tan(2\sqrt{x}) + x\cdot\sec^2(2x^{1/2})\cdot 2\cdot\frac{1}{2}x^{-1/2}$$

$$= \tan(2\sqrt{x}) + x^{1/2}\sec^2(2\sqrt{x})$$

FACTS: $\frac{d}{dx}(\tan x) = \sec^2 x$ or $\sec x \sec x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

each of these is $\sec x$ times the "other" one

Question 3.5. Find the derivative of the function.

$$k(x) = x^2 \sec\left(\frac{1}{x}\right) = x^2 \sec\left(x^{-1}\right)$$

$$k'(x) = \left(x^2\right)' \sec\left(x^{-1}\right) + x^2\left(\sec\left(x^{-1}\right)\right)'$$

$$= 2x \sec\left(x^{-1}\right) + x^2 \cdot \sec\left(x^{-1}\right)\tan\left(x^{-1}\right)\cdot(-1)x^{-2}$$

$$= 2x \sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$$

or $\sec\left(\frac{1}{x}\right)\cdot\left(2x - \tan\left(\frac{1}{x}\right)\right)$

If you want more explanation for one of the steps:

If $f(x) = \sec\left(\frac{1}{x}\right) = \sec\left(x^{-1}\right)$

then $f = \sec u$ and $u = x^{-1}$

$$\frac{df}{du} = \sec u \tan u \qquad \frac{du}{dx} = -x^{-2}$$

so $f'(x) = \frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = \sec u \tan u \cdot (-1)x^{-2}$

$$= \sec\left(x^{-1}\right)\tan\left(x^{-1}\right)\cdot(-1)x^{-2}$$

Question 3.6. Find the derivative of the function.

$$f(x) = \sqrt{7 + x\sec x}$$

$$f(x) = (7 + x\sec x)^{1/2} \quad \text{(COMPOSITION)}$$

$$f'(x) = \underbrace{\frac{1}{2}(7 + x\sec x)^{-1/2}}_{\substack{\text{Took deriv. of outside} \\ \text{Left inside alone}}} \cdot \underbrace{(7 + x\sec x)'}_{\text{Deriv. of inside}}$$

$$= \frac{1}{2}(7 + x\sec x)^{-1/2} \cdot \left(0 + \underbrace{(x)'}_{1}\sec x + x\underbrace{(\sec x)'}_{\sec x \tan x}\right)$$

$$= \frac{1}{2}(7 + x\sec x)^{-1/2} \cdot \underbrace{(\sec x + x\sec x\tan x)}_{\text{Can factor if we like}}$$

OR $$\frac{1}{2} \cdot \frac{1}{\sqrt{7 + x\sec x}} \cdot \sec x \cdot (1 + x\tan x)$$

OR $$\frac{\sec x \cdot (1 + x\tan x)}{2\sqrt{7 + x\sec x}}$$

Question 3.7. Find the derivative of the function.

$$g(x) = \frac{\tan 3x}{(x+7)^4}$$

QUOTIENT

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

composition

$$g'(x) = \frac{(\tan 3x)'(x+7)^4 - \tan(3x)\cdot\left((x+7)^4\right)'}{\left((x+7)^4\right)^2}$$

$$= \frac{\sec^2(3x)\cdot 3\cdot (x+7)^4 - \tan(3x)\cdot 4(x+7)^3\cdot 1}{(x+7)^8}$$

You can stop there, but you should also practice algebra skills. The numerator contains a common factor of $(x+7)^3$

$$= \frac{(x+7)^3\cdot\left[\sec^2(3x)\cdot 3\cdot (x+7) - \tan(3x)\cdot 4\right]}{(x+7)^8}$$

$$= \frac{3(x+7)\sec^2 3x - 4\tan 3x}{(x+7)^5}$$

Question 3.8. Find the derivative of the function.

$$y = 2\ln(\sin x)$$

$$y = 2\ln u \quad \text{and} \quad u = \sin x$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{dy}{du} = 2 \cdot \frac{1}{u} = \frac{2}{u} \qquad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{2}{u} \cdot \cos x$$

$$= \frac{2}{\sin x} \cdot \cos x$$

$$\text{or} \quad 2 \cdot \frac{\cos x}{\sin x} \quad \text{or} \quad 2\cot x \quad \text{or} \quad \frac{2}{\tan x}$$

Question 3.9. Find the derivative of the function.

$$y = \underbrace{\ln(x^3)}_{\text{Composition}} - \underbrace{(\ln x)^3}_{\text{Composition}}$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{x^3}}_{\substack{\text{Took deriv}\\ \text{of outside,}\\ \text{left inside alone}}} \cdot \underbrace{3x^2}_{\substack{\text{Deriv of}\\ \text{inside}}} - \underbrace{3(\ln x)^2}_{\substack{\text{Took deriv}\\ \text{of outside,}\\ \text{left inside alone}}} \cdot \underbrace{\frac{1}{x}}_{\substack{\text{Deriv of}\\ \text{inside}}}$$

$$= \frac{3}{x} - \frac{3(\ln x)^2}{x} \quad \text{or} \quad \frac{3(1-(\ln x)^2)}{x}$$

If you want more explanation:

$$f = \ln(x^3) \Rightarrow f = \ln u \text{ and } u = x^3$$

$$\frac{df}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 3x^2 \qquad \frac{df}{dx} = \frac{1}{u} \cdot 3x^2$$

$$g = (\ln x)^3 \Rightarrow g = u^3 \text{ and } u = \ln x$$

$$\frac{dg}{du} = 3u^2 \qquad \frac{du}{dx} = \frac{1}{x} \qquad \frac{dg}{dx} = 3u^2 \cdot \frac{1}{x}$$

Question 3.10. Find the derivative of the function.

$$y = \frac{x^4}{4} \ln x - \frac{x^4}{16}$$

$$y = \frac{1}{4} \cdot x^4 \cdot \ln x - \frac{1}{16} \cdot x^4$$

$$\frac{dy}{dx} = \frac{1}{4} \cdot \left((x^4)' \ln x + x^4 (\ln x)' \right)' - \frac{1}{16} (x^4)'$$

$$= \frac{1}{4} \left(4x^3 \ln x + x^4 \cdot \frac{1}{x} \right) - \frac{1}{16} \cdot 4x^3$$

$$= x^3 \ln x + \frac{1}{4} x^3 - \frac{1}{4} x^3$$

$$= x^3 \ln x$$

Facts: $\dfrac{d}{dx}\left(\sin^{-1}x\right) = \dfrac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

Question 3.11. Find the derivative of the function.

$$y = \sin^{-1}(1-t)$$

$y = \sin^{-1}(u)$ and $u = 1-t$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \qquad \frac{du}{dt} = -1$$

$$\frac{dy}{dt} = \frac{dy}{du}\cdot\frac{du}{dt} = \frac{1}{\sqrt{1-u^2}}\cdot(-1) = \frac{-1}{\sqrt{1-u^2}}$$

$$= \frac{-1}{\sqrt{1-(1-t)^2}} \quad \text{OR} \quad \frac{-1}{\sqrt{1-(1-2t+t^2)}}$$

$$\text{OR} \quad \frac{-1}{\sqrt{2t-t^2}}$$

Question 3.12. Find the derivative of the function.

$$y = \ln(\tan^{-1} x)$$

$$y = \ln u \quad \text{and} \quad u = \tan^{-1} x$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{dy}{du} = \frac{1}{u} \qquad\qquad \frac{du}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$\text{or} \quad \frac{1}{(1+x^2)\tan^{-1} x}$$

Question 3.13. Find the derivative of the function.

$$y = \tan^{-1}(\ln x)$$

$$y = \tan^{-1} u \quad \text{and} \quad u = \ln x$$

$$\frac{dy}{du} = \frac{1}{1+u^2} \qquad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \cdot \frac{1}{x}$$

$$= \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}$$

$$\text{OR} \quad \frac{1}{x\left(1+(\ln x)^2\right)}$$

Question 3.14. Find the derivative of the function.

$$y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{x^2+4}\cdot(x^2+4)' - \left((x)'\tan^{-1}\left(\frac{x}{2}\right) + x\left(\tan^{-1}\left(\frac{x}{2}\right)\right)'\right)$$

$$= \frac{1}{x^2+4}\cdot 2x - \left(1\tan^{-1}\left(\frac{x}{2}\right) + x\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}\right)$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{x}{\left(1+\frac{x^2}{4}\right)\cdot 2}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{\left(1+\frac{x^2}{4}\right)\cdot 4}$$

$$= \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2}$$

$$= -\tan^{-1}\left(\frac{x}{2}\right)$$

Question 3.15. The following implicitly defines y as a function of x. Find dy/dx.

$$x^2y + xy^2 = 6$$

$$\frac{d}{dx}\left(x^2y + xy^2\right) = \frac{d}{dx}(6)$$

$$(x^2)'y + x^2(y)' + (x)'y^2 + x(y^2)' = 0$$

where $'$ means $\frac{d}{dx}$

Remember: y is not x! y is a function of x!

$$2x \cdot y + x^2 \cdot y' + 1 \cdot y^2 + x \cdot 2y \cdot y' = 0$$

$$x^2 \cdot y' + x \cdot 2y \cdot y' = -2xy - y^2$$

$$(x^2 + 2xy)y' = -2xy - y^2$$

$$y' = \frac{-2xy - y^2}{x^2 + 2xy}$$

OR $\dfrac{-y(2x+y)}{x(x+2y)}$

Must treat y^3 like $(\text{FUNCTION})^3$

Question 3.16. The following implicitly defines y as a function of x. Find dy/dx.

$$x^3 + y^3 = 18xy$$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(18xy\right)$$

$$3x^2 + 3y^2 \cdot y' = 18\left(\underbrace{(x)'}_{1}y + x(y)'\right)$$

$$3x^2 + 3y^2 \cdot y' = 18y + 18x \cdot y'$$

$$3y^2 \cdot y' - 18x \cdot y' = 18y - 3x^2$$

$$\left(3y^2 - 18x\right)y' = 18y - 3x^2$$

$$y' = \frac{18y - 3x^2}{3y^2 - 18x}$$

or $y' = \dfrac{3(6y - x^2)}{3(y^2 - 6x)} = \dfrac{6y - x^2}{y^2 - 6x}$

FACTS: $\frac{d}{dt}(\tan t) = \sec^2 t$ ← each is secant times something

$\frac{d}{dt}(\sec t) = \sec t \tan t$

Question 3.17. The following implicitly defines y as a function of x. Find dy/dx.

$$x = \sec y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sec y)$$

$$1 = \sec y \tan y \cdot y'$$

$$\boxed{\frac{1}{\sec y \tan y} = y'}$$

This counts as an answer.
Can we rewrite it?
We can replace $\sec y$ with x.
Can we replace $\tan y$?

FACTS:

$\sin^2 y + \cos^2 y = 1$

↓ divide by $\cos^2 y$

$\tan^2 y + 1 = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$ So $\tan y = \pm\sqrt{\sec^2 y - 1}$

but we don't know whether $\tan y$ is positive or negative without more information