Linear approximation formula:

$$L(x) = \underbrace{f(a)}_{\text{nice output}} \underbrace{+}_{\text{plus}} \overbrace{\underbrace{f'(a)}_{\text{slope}} \cdot \underbrace{(x-a)}_{\text{change in input}}}^{}$$

change in output

Idea: a = nice input

x = general input that might not be nice

Part 4

Linearization

Question 4.1. Find the linear approximation of $f(x) = \sqrt{x^2+9}$ at $a = -4$.

$$f(x) = (x^2+9)^{1/2}$$

$$f'(x) = \frac{1}{2}(x^2+9)^{-1/2} \cdot 2x$$ using chain rule

$$f(a) = f(-4) = ((-4)^2+9)^{1/2} = (16+9)^{1/2} = 25^{1/2} = 5$$

$$f'(a) = \frac{1}{2}((-4)^2+9)^{-1/2} \cdot 2(-4) = \frac{1}{2}(25)^{-1/2} \cdot (-8)$$

$$= \frac{1}{2} \cdot \frac{1}{5} \cdot (-8) = \frac{-4}{5}$$

So linear approximation is

$$5 + \frac{-4}{5}(x-(-4))$$

or $5 - \frac{4}{5}(x+4)$ or $5 - \frac{4}{5}x - \frac{16}{5}$ or $\frac{9}{5} - \frac{4x}{5}$

$$\text{Linear approximation is } \underbrace{f(a)}_{\text{nice output}} \underbrace{+}_{\text{plus}} \underbrace{\overbrace{f'(a)}^{\text{slope}} \overset{\text{times}}{\cdot} \overbrace{(x-a)}^{\text{change in input}}}_{\text{change in output}}$$

Question 4.2. Use linear approximation to estimate the cube root of 1006.

Key: Wanting the cube root of a number is the same as wanting $x^{1/3}$ for some value of x

Other key: We must "notice" that 1006 is "close" to a "nice" number whose cube root we know. Specifically, 1006 is close to 1000.

So, choose $f(x) = x^{1/3}$ and $a = 1000$.

Then $f'(x) = \frac{1}{3} x^{-2/3}$

$f(a) = f(1000) = 1000^{1/3} = 10$

$$f'(a) = f'(1000) = \frac{1}{3} \cdot 1000^{-2/3} = \frac{1}{3}(1000^{1/3})^{-2}$$
$$= \frac{1}{3} \cdot 10^{-2} = \frac{1}{3} \cdot \frac{1}{100} = \frac{1}{300}$$

So linear approximation is $10 + \frac{1}{300} \cdot (x - 1000)$

So cube root of 1006 (i.e. $f(1006) = 1006^{1/3}$) is approximately

$$10 + \frac{1}{300}(1006 - 1000) = 10 + \frac{1}{300} \cdot 6 = 10 + \frac{1}{50}$$
$$= 10.02$$

Question 4.3. Find the linear approximation of $f(x) = \sqrt{x+1} + \sin x$ at $a = 0$, and use it to estimate $\sqrt{1.004} + \sin(0.004)$.

$$f(x) = (x+1)^{1/2} + \sin x$$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} \cdot 1 + \cos x$$

$$f(a) = f(0) = (0+1)^{1/2} + \sin 0 = 1^{1/2} + 0 = 1$$

$$f'(a) = f'(0) = \frac{1}{2}(0+1)^{-1/2} + \cos 0$$

$$= \frac{1}{2} \cdot 1 + 1 = \frac{3}{2}$$

So linear approximation is $1 + \frac{3}{2} \cdot (x - 0) = 1 + \frac{3}{2}x$

$f(a) + f'(a) \cdot (x-a)$

We want $\sqrt{1.004} + \sin(0.004)$, which is $f(0.004)$

$\sqrt{0.004+1} + \sin(0.004)$

So that's approximately $1 + \frac{3}{2}(0.004)$

$= 1 + 0.006 = 1.006$.