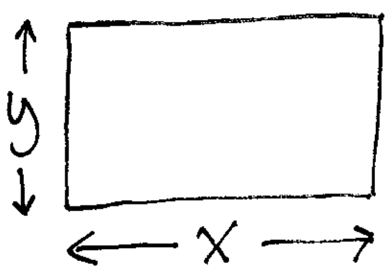
Part 6

Applied Optimization

Question 6.1. What is the smallest perimeter possible for a rectangle whose area is 25 square inches, and what are its dimensions?

Draw a picture and make up variable names!



WANT to minimize perimeter.

Perimeter $= P = 2x + 2y$

GIVEN: Area $= 25$

and we know area $= xy$, so $xy = 25$

$$\Rightarrow y = \frac{25}{x}. \text{ Then } P = 2x + 2\cdot\frac{25}{x} = \underbrace{2x + 50x^{-1}}$$

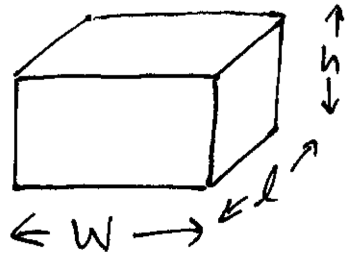
THIS IS THE FUNCTION WE WANT TO MINIMIZE

$$\Rightarrow P'(x) = 2 - 50x^{-2} = 2 - \frac{50}{x^2}.$$

Critical points? $P' = 0 \Rightarrow 2 = \frac{50}{x^2} \Rightarrow x^2 = 25 \Rightarrow x = 5$

So probably $x = 5$ gives the minimum. To confirm, look at where P' is pos. or neg. $x = 1 \Rightarrow P' < 0$, $x = 1000 \Rightarrow P' > 0$, so $x = 5$ really does give a minimum. The rectangle should be 5 inches by 5 inches.

FUNDAMENTAL FACT:

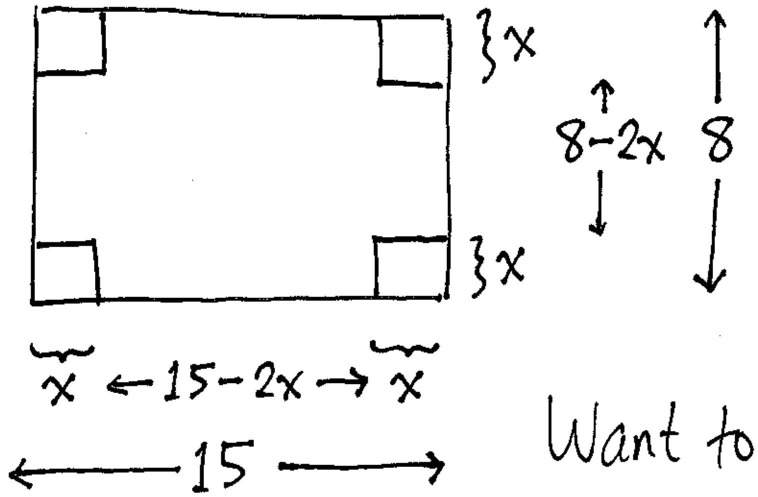


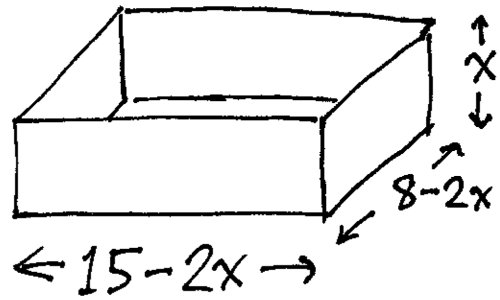
Volume $= w \cdot l \cdot h$

MUST KNOW THIS!

Question 6.2. You are planning to make an open rectangular box from an 8 inch by 15 inch piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of <u>largest volume</u> you can make this way, and what is its volume?

Draw picture and make up variable names!





Want to MAXIMIZE VOLUME

$$\text{Volume} = V = x(8-2x)(15-2x)$$

THIS IS THE FUNCTION WE WANT TO MAXIMIZE

$$= x(120 - 46x + 4x^2) = 120x - 46x^2 + 4x^3$$

$$V' = 120 - 92x + 12x^2 = 4(\underbrace{30 - 23x + 3x^2}_{\text{Can we factor?}})$$

Try ac method. $ac = 90$

$90 = 2 \cdot 45$ or $3 \cdot 30$ or $5 \cdot 18$

$$= 4(30 - 5x - 18x + 3x^2)$$

$$= 4(5(6-x) - 3x(6-x)) = 4(5-3x)(6-x)$$

Critical points? $V' = 0$? $3x = 5$ or $x = 6$

i.e. $x = \frac{5}{3}$ or $x = 6$

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Domain? Bounds on x? Notice from our picture that x must be between 0 and 4.
(Can't cut away more than half the width.)

So, we can ignore $x=6$. The only critical point in our domain is $x = \frac{5}{3}$.

If $x = \frac{5}{3}$, then volume is $x(8-2x)(15-2x)$

$$= \frac{5}{3}\left(8 - \frac{10}{3}\right)\left(15 - \frac{10}{3}\right) = \frac{5}{3}\left(\frac{24}{3} - \frac{10}{3}\right)\left(\frac{45}{3} - \frac{10}{3}\right)$$

$$= \frac{5}{3} \cdot \frac{14}{3} \cdot \frac{35}{3} = \frac{70 \cdot 35}{27} = \frac{2450}{27}.$$

If $x=0$ then volume $=0$

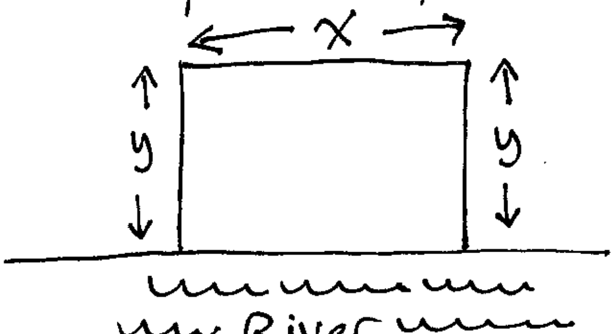
If $x=4$ then volume $=0$

So $x = \frac{5}{3}$ really does give <u>maximum</u> volume.

The dimensions of the box are $\frac{5}{3}$ by $\frac{14}{3}$ by $\frac{35}{3}$ and the maximum volume is $\frac{2450}{27}$.

Question 6.3. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

Interpret the problem! Draw a picture, make up variable names.



Only three sides use fence so we're probably assuming only three sides use wire.

GIVEN: Total length of fence/wire is 800, so $x + 2y = 800$

$\Rightarrow x = 800 - 2y$

WANT to maximize the area. $A = xy$

For variety, let's have the independent variable be y this time.

$A = xy = (800 - 2y)\, y = 800y - 2y^2.$

THIS IS THE FUNCTION WE WANT TO MAXIMIZE.

$A' = A'(y) = \frac{dA}{dy} = 800 - 4y$. Critical points?

$A' = 0 \Rightarrow 4y = 800$

$y = 200$

$y = 200$ is the only critical number, so this probably gives a max.

To confirm, check A'. If $y = 1$, then $A' = 800 - 4y$ is positive.

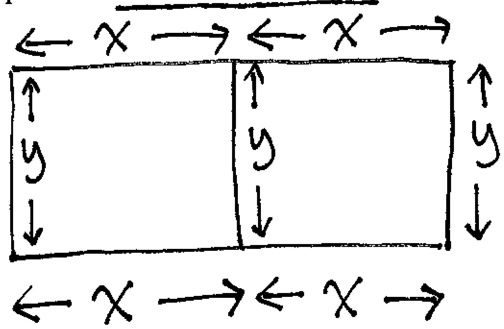
If $y = 1000$, then $A' = 800 - 4y$ is negative.

So A really is increasing before the critical number and decreasing after.

Plot should be 200 by 400 and area is 80000 square meters.

Question 6.4. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Interpret the problem
Draw a picture
Make up variable names $\Rightarrow$



GIVEN: Total area $= 216$, so $2x \cdot y = 216 \Rightarrow xy = 108$

$$y = \frac{108}{x}$$

Want to MINIMIZE total length of fence

Total length of fence $= L = 4x + 3y$

$$L = 4x + 3 \cdot \frac{108}{x} = 4x + 324x^{-1}$$ THIS IS THE FUNCTION WE WANT TO MINIMIZE

$$\frac{dL}{dx} = 4 - 324x^{-2} = 4 - \frac{324}{x^2}$$ Critical numbers?

$$\frac{dL}{dx} = 0 \Rightarrow 4 = \frac{324}{x^2} \Rightarrow 4x^2 = 324 \Rightarrow x^2 = 81 \Rightarrow x = 9$$

(must be positive)

We suspect $x = 9$ probably gives the minimum value of L.

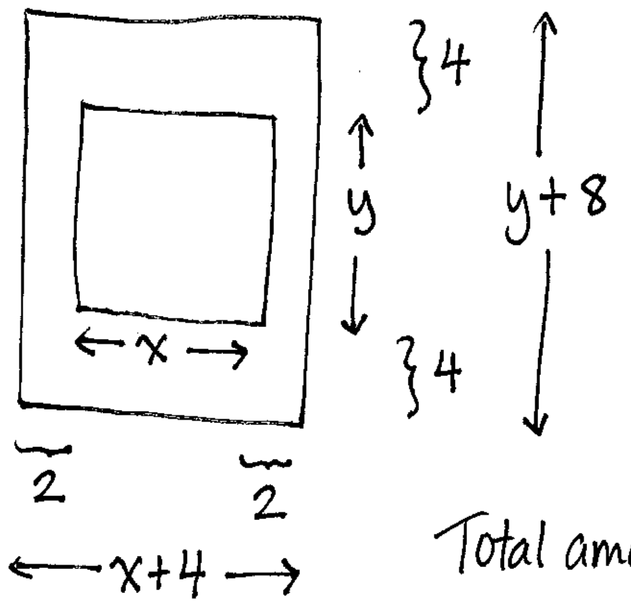
To confirm, look at L'. If $x = 1$ then $L' = 4 - \frac{324}{x^2}$ is negative

If $x = 100$ then $L' = 4 - \frac{324}{x^2}$ is positive

So L really is decreasing before the critical point and increasing after, so $x = 9$ gives minimum. Outer rectangle is 18 by 12 and total length $= 36 + 36 = 72$.

Question 6.5. You are designing a rectangular poster to contain 50 square inches of printing with a 4-inch margin at the top and bottom and a 2-inch margin at each side. What overall dimensions will minimize the amount of paper used?

Read, draw a picture, and make up variable names.



GIVEN: 50 square inches of printing

So $xy = 50 \Rightarrow y = \frac{50}{x}$

Want to MINIMIZE the total amount of paper.

Total amount of paper $= A = (x+4)(y+8)$.

$$A = (x+4)\left(\frac{50}{x}+8\right) = 50 + 8x + \frac{200}{x} + 32$$

$$A = 82 + 8x + 200x^{-1}$$ THIS IS THE FUNCTION WE WANT TO MINIMIZE

$$A' = \frac{dA}{dx} = 0 + 8 - 200x^{-2} = 8 - \frac{200}{x^2}$$

Critical numbers? $A' = 0 \Rightarrow 8 = \frac{200}{x^2} \Rightarrow x^2 = 25$

$x = 5$

So $x = 5$ probably minimizes the total amount of paper.

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$$A' = \frac{dA}{dx} = 8 - \frac{200}{x^2}$$

If x is small (e.g. $x=1$) then A' is negative

If x is big (e.g. $x=100$) then A' is positive

So here's what happens on either side of the critical point

$x=5$

A' is negative	A' is positive
A is DEcreasing	A is INcreasing

So, A really is MINIMIZED when $x=5$.

The width of the poster should be $x+4 = 5+4 = 9$ inches

and the height of the poster should be

$$y+8 = \frac{50}{x}+8 = \frac{50}{5}+8 = 10+8 = 18 \text{ inches.}$$