Rule: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

if n is any constant EXCEPT -1.

Part 7

Integrals

We're also using the "sum rule" and the "constant multiple rule".

Question 7.1. Find the integral.

$$\int (2x^3 - 5x + 7) \, dx$$

$$2 \cdot \frac{x^4}{4} - 5 \cdot \frac{x^2}{2} + 7x + C$$

$$= \frac{x^4}{2} - \frac{5x^2}{2} + 7x + C$$

Question 7.2. Find the integral.

$$\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right)\,dx$$

$$= \int \left(\frac{1}{5} - 2x^{-3} + 2x\right) dx$$

$$= \frac{1}{5}x - 2\cdot\frac{x^{-3+1}}{-3+1} + 2\cdot\frac{x^2}{2} + C$$

$$= \frac{x}{5} - 2\cdot\frac{x^{-2}}{-2} + x^2 + C$$

$$= \frac{x}{5} + x^{-2} + x^2 + C$$

OR $\frac{x}{5} + \frac{1}{x^2} + x^2 + C$

or $\frac{x^3}{5x^2} + \frac{5}{5x^2} + \frac{5x^4}{5x^2} + c$

$= \frac{x^3 + 5 + 5x^4}{5x^2} + c$

Question 7.3. Find the integral.

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$$

$$= \int \left(\frac{1}{2} x^{1/2} + 2x^{-1/2} \right) dx$$

$$= \frac{1}{2} \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2 \cdot \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{1}{2} \cdot \frac{x^{3/2}}{3/2} + 2 \cdot \frac{x^{1/2}}{1/2} + C$$

$$= \frac{1}{2} \cdot \frac{2}{3} x^{3/2} + 2 \cdot \frac{2}{1} x^{1/2} + C$$

$$= \frac{1}{3} x^{3/2} + 4x^{1/2} + C$$

Question 7.4. Find the integral.

$$\int 2x(1 - x^{-3})\, dx$$

We don't have a "product rule" for integrals, but we can do some algebraic rearranging before we integrate.

$$\int \left(2x \cdot 1 - 2x \cdot x^{-3}\right) dx$$

$$= \int \left(2x - 2x^{-2}\right) dx$$

$-2+1=-1$

$$= 2 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^{-1}}{-1} + C$$

$$= x^2 + 2x^{-1} + C$$

or $x^2 + \frac{2}{x} + C$

Question 7.5. Find the integral.

$$\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt$$

Again, we can do algebra before we integrate.

$$\int \frac{t \cdot t^{1/2} + t^{1/2}}{t^2}\,dt = \int \frac{t^{3/2}+t^{1/2}}{t^2}\,dt$$

$$= \int \left(\frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2} \right) dt$$

$$\frac{3}{2} - 2 = \frac{3}{2} - \frac{4}{2} = \frac{-1}{2}$$

$$\frac{1}{2} - 2 = \frac{1}{2} - \frac{4}{2} = -\frac{3}{2}$$

$$= \int \left(t^{-1/2} + t^{-3/2} \right) dt$$

$$-\frac{1}{2} + 1 = \frac{1}{2}$$

$$-\frac{3}{2} + 1 = -\frac{1}{2}$$

$$= \frac{t^{1/2}}{1/2} + \frac{t^{-1/2}}{-1/2} + C$$

$$= 2t^{1/2} - 2t^{-1/2} + C$$

or $2\sqrt{t} - \dfrac{2}{\sqrt{t}} + C$

or $\dfrac{2t}{\sqrt{t}} - \dfrac{2}{\sqrt{t}} + c$

$= \dfrac{2(t-1)}{\sqrt{t}} + C$

Question 7.6. Find the integral.

$$\int (4\sec x \tan x - 2\sec^2 x)\, dx$$

FACTS: $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

If we know a derivative fact, then we know an antiderivative fact.

So, FACTS: $\int \sec^2 x \, dx = \tan x + C$

$\int \sec x \tan x \, dx = \sec x + C$

(We just "know" those holistically. There are no "steps".)

So, for this problem, we can say

$$\int (4\sec x \tan x - 2\sec^2 x)\, dx$$

$$= 4\int \sec x \tan x \, dx - 2\int \sec^2 x \, dx$$

$$= \underline{4\sec x - 2\tan x + C}.$$

Question 7.7. Find the integral.

$$\int \left(\frac{1}{x} - \frac{5}{x^2+1}\right) dx$$

FACTS: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ Also $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ or $\frac{1}{x^2+1}$

So we just "know" $\int \frac{1}{x}\, dx = \ln|x| + C$

and $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$

So in the given problem, we can say

$$\int \left(\frac{1}{x} - 5 \cdot \frac{1}{x^2+1} \right) dx$$

$$= \int \frac{1}{x}\, dx - 5 \int \frac{1}{x^2+1}\, dx$$

$$= \underline{\ln|x| - 5\tan^{-1} x + C}$$

Question 7.8. Find the integral.

$$\int \left(\frac{2}{\sqrt{1-y^2}} - \frac{1}{y^{1/4}}\right) dy$$

FACT: $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ so $\frac{d}{dy}\left(\sin^{-1} y\right) = \frac{1}{\sqrt{1-y^2}}$

Therefore, FACT: $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x$

(and similarly if the variable has another name)

So, in the given problem, we have

$$\int \left(2 \cdot \frac{1}{\sqrt{1-y^2}} - y^{-1/4}\right) dy$$

$$= 2 \cdot \sin^{-1} y - \frac{y^{3/4}}{3/4} + C$$

$$= \underline{2 \sin^{-1} y - \frac{4}{3} y^{3/4} + C}$$

Question 7.9. Find the integral.

$$\int 2x(x^2+5)^{-4}\,dx$$

Can't obviously "do algebra" because of the -4 exponent.

Maybe SUBSTITUTION. Try $u = x^2+5$ ("inside" function)

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$\text{Integral} = \int \underbrace{(x^2+5)}_{=u}{}^{-4}\cdot \underbrace{2x\cdot dx}_{=du}$$

$$= \int u^{-4}\,du = \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{3u^3} + C = \underline{-\frac{1}{3(x^2+5)^3} + C}$$

Question 7.10. Find the integral.

$$\int \frac{4x^3}{(x^4+1)^2}\,dx$$

Try substitution. $u = x^4 + 1$ ("inside" function)

$$\downarrow$$

$$\frac{du}{dx} = 4x^3$$

$$du = 4x^3\,dx$$

So integral $= \displaystyle\int \frac{1}{(\underbrace{x^4+1}_{=u})^2} \cdot \underbrace{4x^3\,dx}_{=du}$

$$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{x^4+1} + C$$

Question 7.11. Find the integral.

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\, dx$$

Try substitution. $u = 1+\sqrt{x}$ ("inside" function)

$$u = 1 + x^{1/2}$$

$$\downarrow$$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

$$du = \frac{1}{2}x^{-1/2}dx$$

$$2du = x^{-1/2}dx = \frac{1}{\sqrt{x}}\, dx$$

$$\text{Integral} = \int (\underbrace{1+\sqrt{x}}_{u})^{1/3} \cdot \underbrace{\frac{1}{\sqrt{x}}\, dx}_{=2du} = \int u^{1/3} \cdot 2\, du$$

$$= 2\int u^{1/3} du = 2 \cdot \frac{u^{4/3}}{4/3} + C$$

$$= 2 \cdot \frac{3}{4} u^{4/3} + C = \frac{3}{2} u^{4/3} + C = \underline{\frac{3}{2}(1+\sqrt{x})^{4/3} + C}$$

Question 7.12. Find the integral.

$$\int \sin 3x \, dx = \int \sin(3x) \cdot dx$$

Can do substitution. Try $u = 3x$ ("inside" function)

$$\frac{du}{dx} = 3$$

$$du = 3\,dx$$

$$\frac{1}{3} du = dx$$

$$\text{Integral} = \int \sin\underbrace{(3x)}_{=u} \cdot \underbrace{dx}_{=\frac{1}{3}du} = \int \sin(u) \cdot \frac{1}{3} du$$

$$= \frac{1}{3} \int \sin u \, du = -\frac{1}{3} \cos u + C$$

$$= -\frac{1}{3} \cos 3x + C.$$

(With enough experience, might you be able to "guess" that answer?)

Question 7.13. Find the integral.

$$\int \frac{1}{\sqrt{5x+8}}\,dx = \int (5x+8)^{-1/2}\,dx$$

Try substitution. $u = 5x+8$ ("inside" function)

$$\frac{du}{dx} = 5$$

$$du = 5\,dx$$

$$\frac{1}{5}du = dx$$

Integral $= \int (5x+8)^{-1/2} \cdot dx = \int u^{-1/2} \cdot \frac{1}{5}\,du$

$$= \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5} \cdot \frac{u^{1/2}}{1/2} + c$$

$$= \frac{2}{5}u^{1/2} + C = \frac{2}{5}(5x+8)^{1/2} + C$$

or $\frac{2}{5}\sqrt{5x+8} + C$

Don't forget: We know $\frac{d}{dx}(\tan x) = \sec^2 x$

So we also just know $\int \sec^2 x \, dx = \tan x + C$

Question 7.14. Find the integral.

$$\int \sec^2(3x+2)\,dx$$

Try substitution. $u = 3x+2$ ("inside" function)

$$\frac{du}{dx} = 3$$

$$du = 3\,dx$$

$$\frac{1}{3}du = dx$$

Integral $= \int \sec^2(3x+2)\cdot dx = \int \sec^2(u)\cdot \frac{1}{3}\,du$

$$= \frac{1}{3}\int \sec^2 u \, du = \frac{1}{3}\tan u + C$$

$$= \frac{1}{3}\tan(3x+2) + C$$

Note: $e^{\sin x}$ is of the form $e^{\text{something}}$ or $e^{(\text{not just } x)}$

So $\sin x$ is like the "inside" function

Also, we see the derivative of $\sin x$ in the integral

Question 7.15. Find the integral.

$$\int (\cos x)e^{\sin x}\,dx$$

Try substitution: $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x\,dx$$

$$\text{Integral} = \int e^{\sin x} \cdot \cos x\,dx = \int e^{u} \cdot du$$

$$= e^{u} + C = e^{\sin x} + C$$

(Now that you have some experience, could you have guessed that answer by thinking about the chain rule in reverse?)

Fact: $\int \frac{1}{t}\,dt = \ln|t| + c$ because $\ln|t|$ is defined for positive or negative t and has derivative equal to $\frac{1}{t}$

Question 7.16. Find the integral.

$$\int \frac{1}{x \ln x}\,dx = \int \frac{1}{x} \cdot \frac{1}{\ln x}\,dx$$

What substitution can we try?

Maybe $u = \ln x$ because we also see the derivative of $\ln x$.

$$\downarrow$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\,dx$$

Integral $= \int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx = \int \frac{1}{u} \cdot du$

$= \ln|u| + C = \ln|\ln x| + C$

Question 7.17. A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 minute later?

Acceleration: $a = a(t) = 20$

$\downarrow$ take antiderivative with respect to t

(velocity) $\quad v = v(t) = \int 20\,dt = 20t + c_1$

Assuming the rocket is stationary when we start, we have $v(0) = 0$. But also $v(0) = 20 \cdot 0 + c_1 = c_1$.

So $c_1 = 0$. $\quad v = v(t) = 20t$

$\downarrow$ take antiderivative <u>if</u> we need position

Question asked for velocity 1 <u>minute</u> after we start. But acceleration was in m/sec^2, so time is in seconds. 1 minute $= 60$ seconds. $\quad t = 60$

Velocity after 1 minute is $v(60) = 20 \cdot 60 = 1200$

Question 7.18. A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t})$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$.

(a) Find the velocity $v = ds/dt$ in terms of t.

(b) Find the position s in terms of t.

acceleration

$$a(t) = 15t^{1/2} - 3t^{-1/2}$$

↓ take antiderivative

velocity

$$v(t) = 15 \cdot \frac{t^{3/2}}{3/2} - 3 \cdot \frac{t^{1/2}}{1/2} + C_1$$

$$= 15 \cdot \frac{2}{3} t^{3/2} - 3 \cdot 2 t^{1/2} + C_1$$

$$= 10t^{3/2} - 6t^{1/2} + C_1$$

Given: $\frac{ds}{dt} = 4$ when $t = 1$. That is, $v = 4$ when $t = 1$.

i.e. $v(1) = 4$

$$v(1) = 10 \cdot \underbrace{1^{3/2}}_{=1} - 6 \cdot \underbrace{1^{1/2}}_{=1} + C_1 = 10 - 6 + C_1 = 4 + C_1$$

So $C_1 = 0$

$$\boxed{v(t) = 10t^{3/2} - 6t^{1/2}}$$ ← answer for (a)

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velocity
$$v(t) = 10t^{3/2} - 6t^{1/2}$$

$\downarrow$ take antiderivative

position
$$s(t) = 10 \cdot \frac{t^{5/2}}{5/2} - 6 \cdot \frac{t^{3/2}}{3/2} + C_2$$

$$= 10 \cdot \frac{2}{5} t^{5/2} - 6 \cdot \frac{2}{3} t^{3/2} + C_2$$

$$= 4t^{5/2} - 4t^{3/2} + C_2$$

Given: $s = 0$ when $t = 1$. That is, $s(1) = 0$.

$$s(1) = 4 \cdot \underbrace{1^{5/2}}_{=1} - 4 \cdot \underbrace{1^{3/2}}_{=1} + C_2 = 4 - 4 + C_2$$

$$= C_2$$

So $C_2 = 0$

$$\boxed{s(t) = 4t^{5/2} - 4t^{3/2}}$$ ← answer for (b)