8.1 $f(x) = x^2 - 4x + 3$

(a) Domain: all x (because f is a polynomial)

x-intercepts $\Rightarrow$ $y=0$ $\Rightarrow$ $x^2-4x+3=0$

$(x-1)(x-3)=0$

$x=1$ or $x=3$

Points $(1,0)$ $(3,0)$

y-intercept $\Rightarrow$ $x=0$ $\Rightarrow$ $y=f(0)=0^2-4\cdot 0+3=3$

Point $(0,3)$

(b) $f'(x) = 2x-4 = 2(x-2)$

Critical points: $x=2$

$x=2$

$f'(x)=2(x-2)$	$f'(x)=2(x-2)$
f' is NEGative	f' is POSitive
(e.g. plug in $x=0$ or 1)	(e.g. plug in $x=3$ or 100)
f is DEcreasing	f is INcreasing

local MIN at $x=2$

8.1 continued

(c) $f''(x) = 2$

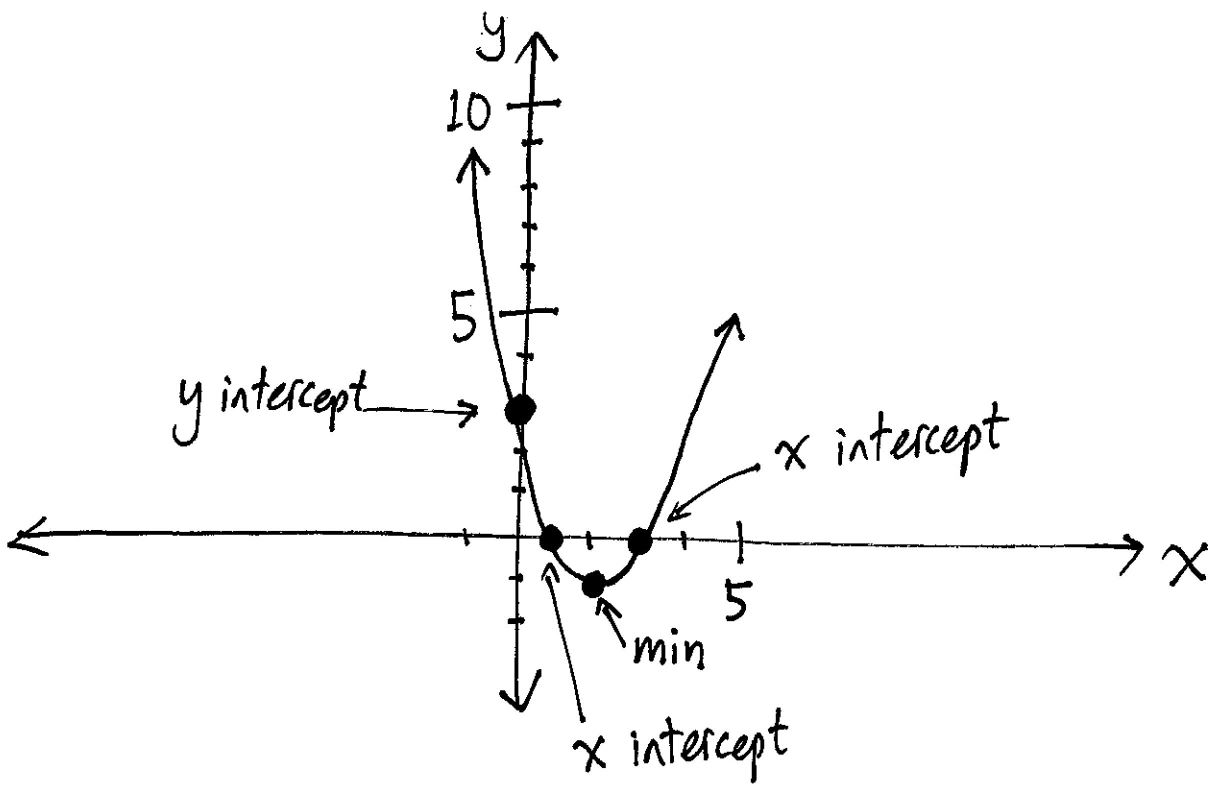
f'' is always positive. There are no inflection points.

f is always concave UP.

(d)

Some values

x	$f(x) = x^2 - 4x + 3 = (x-1)(x-3)$
-1	$(-2)(-4) = 8$
0	$(-1)(-3) = 3$
1	$(0)(-2) = 0$
2	$(1)(-1) = -1$
3	$(2)(0) = 0$
4	$(3)(1) = 3$



8.2 $f(x) = x^3 - 3x + 3$

(a) Domain: all x (because x is a polynomial)

x-intercepts $\Rightarrow y=0 \Rightarrow x^3 - 3x + 3 = 0$

We <u>can't</u> factor or solve most cubics. But instead, maybe can estimate where an x-intercept is:

$$f(0) = 0^3 - 3\cdot 0 + 3 = 3$$
$$f(-1) = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5$$
$$f(-2) = (-2)^3 - 3(-2) + 3 = -8 + 6 + 3 = 1$$
$$f(-3) = (-3)^3 - 3(-3) + 3 = -27 + 9 + 3 = -15$$

So there must be an x-intercept between $x = -2$ and $x = -3$.

y-intercept $\Rightarrow x = 0 \Rightarrow y = f(0) = 0^3 - 3\cdot 0 + 3 = 3$

Point $(0,3)$

(b) $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$= 3(x+1)(x-1)$

Critical points: $x = -1$, $x = 1$

		$x=-1$		$x=1$	
$x+1$	$- \; - \; -$	0	$+ \; + \; +$		$+ \; + \; +$
$x-1$	$- \; - \; -$		$- \; - \; -$	0	$+ \; + \; +$
f'	pos		neg		pos
	f INcreasing		f DEcreasing		f INcreasing
		local MAX at $x=-1$		local MIN at $x=1$	

8.2 continued

(c) $f''(x) = 6x$

$f'' = 0$ if $x = 0$

$x = 0$

f'' is NEGative f is concave DOWN	f'' is POSitive f is concave UP

inflection point at $x=0$

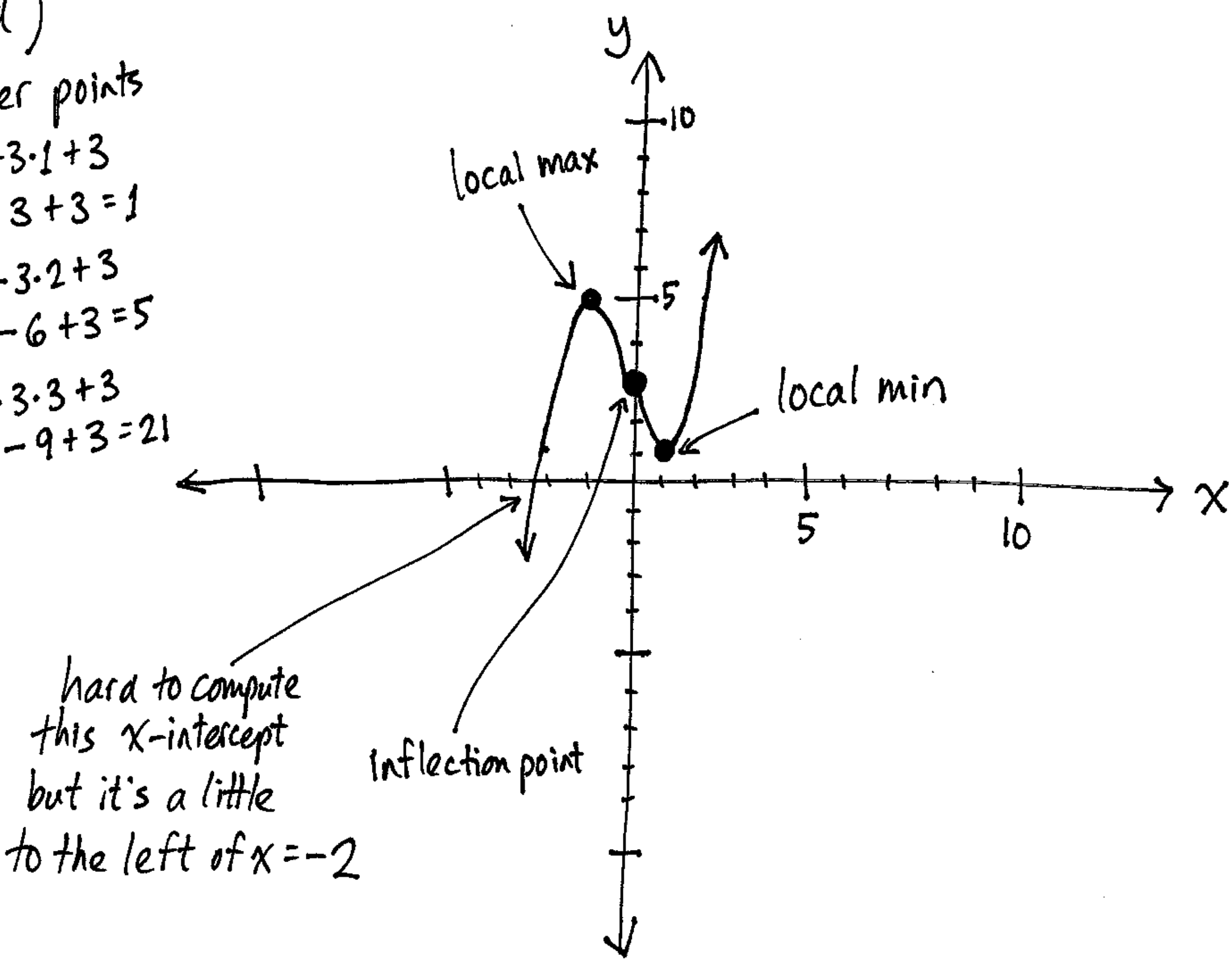
(d)

Some other points

$f(1) = 1^3 - 3\cdot 1 + 3$
$= 1 - 3 + 3 = 1$

$f(2) = 2^3 - 3\cdot 2 + 3$
$= 8 - 6 + 3 = 5$

$f(3) = 3^3 - 3\cdot 3 + 3$
$= 27 - 9 + 3 = 21$



8.3 $f(x) = x^4 - 2x^2$

(a) Domain: all x (because f is a polynomial)

x-intercepts $\Rightarrow y=0 \Rightarrow x^4 - 2x^2 = 0$

$$x^2(x^2-2) = 0$$

$\Rightarrow x^2 = 0$ or $x^2 = 2$

$\Rightarrow x = 0$ or $x = \sqrt{2}$ or $x = -\sqrt{2}$. Points: $(-\sqrt{2}, 0)$, $(0,0)$, $(\sqrt{2}, 0)$

y-intercept $\Rightarrow x = 0 \Rightarrow y = f(0) = 0^4 - 2\cdot 0^2 = 0$

Point: $(0,0)$

(b) $f'(x) = 4x^3 - 4x$

$$= 4x(x^2-1) = 4x(x+1)(x-1)$$

Critical points: $x=0$, $x=-1$, $x=1$

		$x=-1$		$x=0$		$x=1$	
$4x$	— — —		— — —	0	+ + +		+ + +
$x+1$	— — —	0	+ + +		+ + +		+ + +
$x-1$	— — —		— — —		— — —	0	+ + +
f'	f' negative f DEcreasing		f' positive f INcreasing		f' negative f DEcreasing		f' positive f INcreasing
		local MIN at $x=-1$		local MAX at $x=0$		local MIN at $x=1$	

8.3 continued

(c) $f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$

$f'' = 0 \Rightarrow 3x^2 = 1 \Rightarrow x^2 = 1/3 \Rightarrow x = \pm 1/\sqrt{3}$

or $x = \pm\sqrt{3}/3$

It may help to know $\sqrt{3} \approx 1.7$

So $\sqrt{3}/3$ is a bit less than 0.6.

$x = -\sqrt{3}/3$ $\qquad\qquad$ $x = \sqrt{3}/3$

Test value $x = -100$	Test value $x = 0$	Test value $x = 100$
$f'' = 4(30000 - 1)$	$f'' = 4(0 - 1)$	$f'' = 4(30000 - 1)$
f'' positive	f'' NEGative	f'' positive
f concave UP	f concave DOWN	f concave UP

inflection point $\qquad\qquad$ inflection point

(d) For graphing purposes, we should evaluate f at all "special" points. $f(x) = x^4 - 2x^2 = x^2(x^2 - 2)$

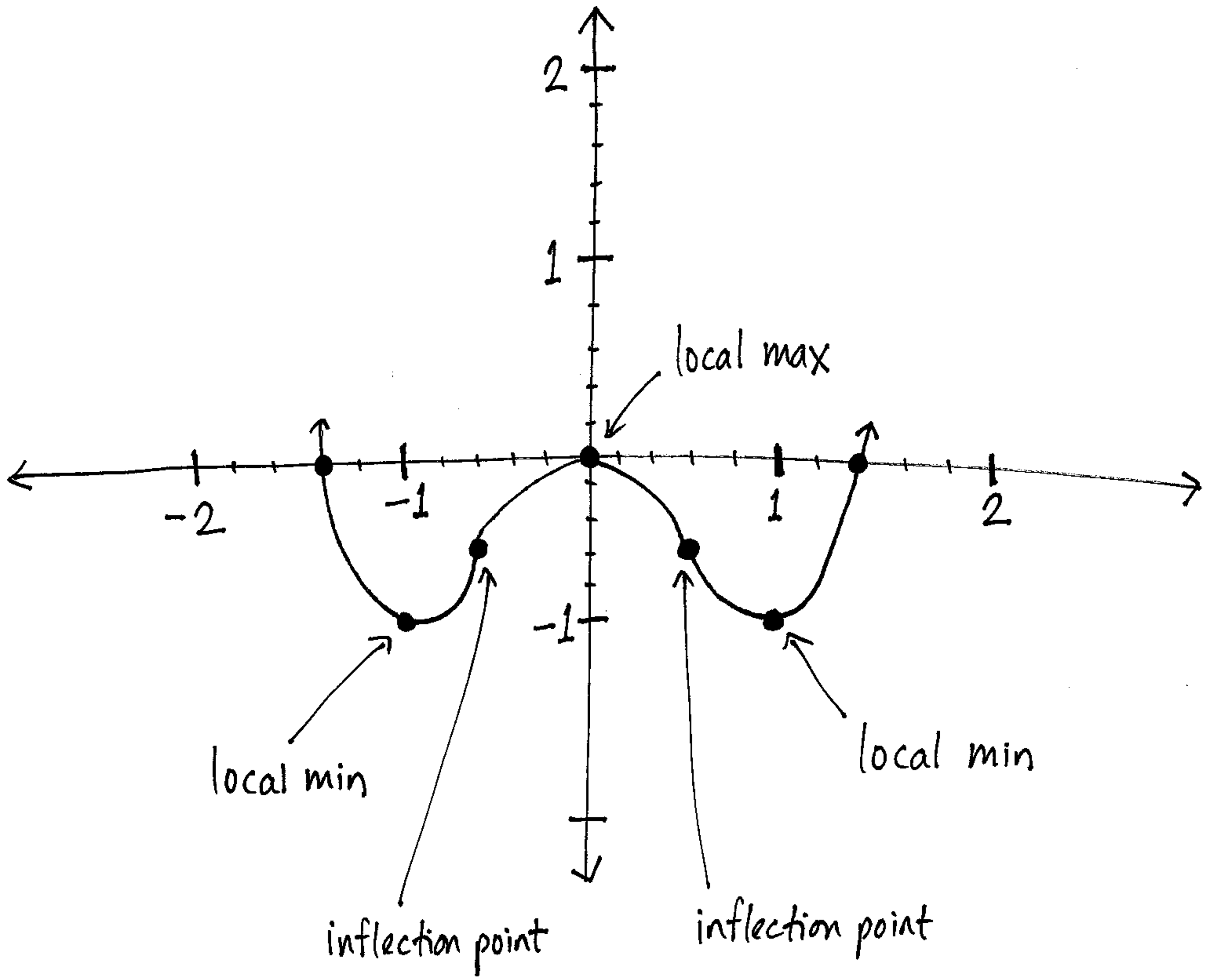
$f(0) = 0^2(0^2 - 2) = 0$

$f(\pm 1/\sqrt{3}) = \frac{1}{3} \cdot \left(\frac{1}{3} - 2\right) = \frac{1}{3} \cdot \frac{-5}{3} = \frac{-5}{9}$ slightly less than $-\frac{1}{2}$

$f(\pm 1) = 1 \cdot (1 - 2) = -1$

$f(\pm\sqrt{2}) = 2 \cdot (2 - 2) = 0$

8.3 continued



8.4 $f(x) = x + \sin x$

(a) Domain: all x (polynomials and sine functions have no restrictions on the domain)

x-intercepts $\Rightarrow y=0 \Rightarrow x + \sin x = 0$

or $\sin x = -x$

Can't solve this algebraically. Can we guess intercepts? Find them graphically?

y-intercepts $\Rightarrow x=0 \Rightarrow y = 0 + \sin 0 = 0 + 0 = 0$

Point $(0,0)$

(both an x-intercept and y-intercept)

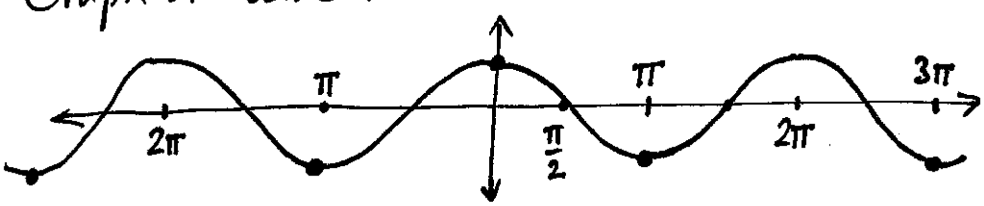
(b) $f'(x) = 1 + \cos x$

Critical points? f' is never undefined

$f'=0 \Rightarrow 1+\cos x = 0 \Rightarrow \cos x = -1$

You should be able to figure out what x values make $\cos x = -1$. Answer: $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

(How? Graph of cosine function looks like



)

Next, when is f' positive or negative?

Fact: We know $\cos x \geq -1$ always. So here, $f'(x) = 1 + \cos x$ is ≥ 0 always. f' is never negative.

So f is not decreasing on any interval.

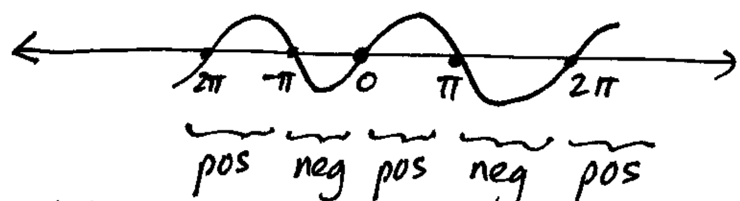
None of the critical points are maxes or mins.

(c) $f''(x) = 0 - \sin x = -\sin x$

$f'' = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

(we know this from KNOWING the sine function)

We know $\sin x$ behaves like



so $f''(x) = -\sin x$ behaves like:

	$x=-3\pi$		$x=-2\pi$		$x=-\pi$		$x=0$		$x=\pi$		$x=2\pi$		$x=3\pi$	
f'' neg		f'' pos		f'' neg		f'' pos		f'' neg		f'' pos		f'' neg		f'' pos
$\frown$		$\smile$		$\frown$		$\smile$		$\frown$		$\smile$		$\frown$		$\smile$

Each of the points $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ is an inflection pt.

(d) It may be helpful to notice

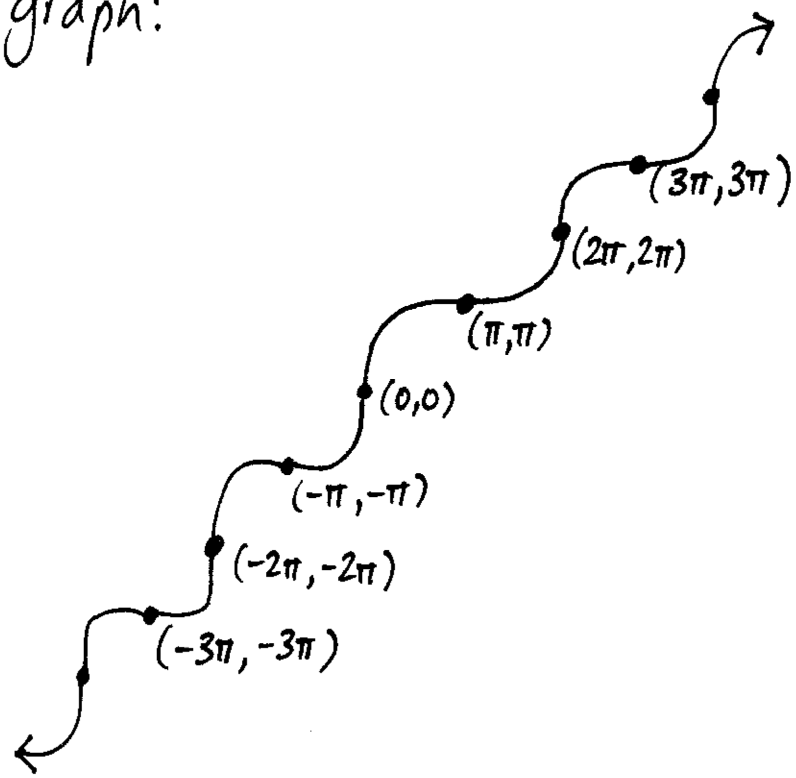
$$f(0) = 0 + \sin 0 = 0$$
$$f(\pi) = \pi + \sin \pi = \pi + 0 = \pi$$
$$f(2\pi) = 2\pi + \sin 2\pi = 2\pi + 0 = 2\pi$$

etc.

Similarly $f(-\pi) = -\pi$, $f(-2\pi) = -2\pi$, etc.

So f is never decreasing,
and changes concavity at each multiple of π.

Rough graph:



For a more careful graph, we could note that
$f'(x) = 1 + \cos x$ is always between 0 and 2.
So steepest slope should be 2 (not vertical).

8.5 $f(x) = \sqrt{16-x^2} = (16-x^2)^{1/2}$

(a) Domain: must have $16-x^2 \geq 0$

$\Rightarrow 16 \geq x^2$ or equivalently $x^2 \leq 16$

$\Rightarrow -4 \leq x \leq 4$

x-intercepts $\Rightarrow y=0 \Rightarrow \sqrt{16-x^2}=0$

$16-x^2=0$

$16 = x^2$

$x = \pm 4$ Points $(4,0)$

$(-4,0)$

y-intercepts $\Rightarrow x=0 \Rightarrow y=\sqrt{16-0^2}=\sqrt{16}=4$

Point $(0,4)$

(b) $f'(x) = \frac{1}{2}(16-x^2)^{-1/2} \cdot (-2x)$

$= \dfrac{-x}{\sqrt{16-x^2}}$ Critical numbers?

When is f' zero or undefined?

When top or bottom is zero.

$-x=0 \Rightarrow x=0$

$\sqrt{16-x^2}=0 \Rightarrow x=4$ or -4 $\qquad -4, 0, 4$

	$x=-4$	$x=0$	$x=4$
$-x$	$+\;+\;+$	0	$-\;-\;-$
$\sqrt{16-x^2}$	$+\;+\;+$		$+\;+\;+$
	f' is positive f is increasing		f' is negative f is decreasing

$\uparrow$

Local max at $x=0$

Technically also a min at $x=-4$ and at $x=4$

(c) since $f'(x) = \dfrac{-x}{(16-x^2)^{1/2}}$

we have
$$f''(x) = \frac{(-x)'(16-x^2)^{1/2} - (-x)\left((16-x^2)^{1/2}\right)'}{\left((16-x^2)^{1/2}\right)^2}$$

$$= \frac{-1(16-x^2)^{1/2} + x \cdot \frac{1}{2}(16-x^2)^{-1/2} \cdot (-2x)}{16-x^2}$$

$$= \frac{-1(16-x^2)^{1/2} - x^2(16-x^2)^{-1/2}}{16-x^2}$$

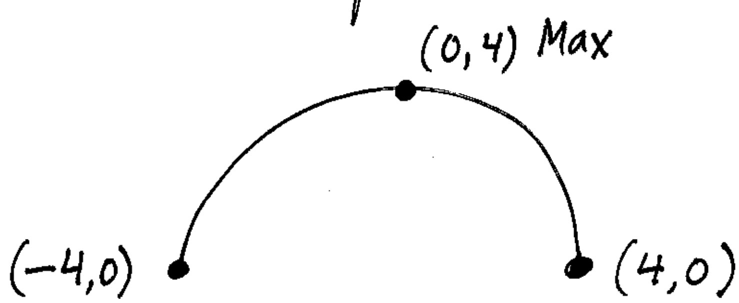
$$= \frac{-(16-x^2)^{-1/2}\left[(16-x^2) + x^2\right]}{16-x^2}$$

The top had two powers of $16-x^2$ and $\frac{-1}{2}$ was the smaller exponent

$$= \frac{-16(16-x^2)^{-1/2}}{(16-x^2)^1} = \frac{-16}{(16-x^2)^{3/2}}$$

So f'' is always negative so f is always concave down. No inflection points.

(d)



NOTE: $y = \sqrt{16-x^2} \Rightarrow y^2 = 16 - x^2$

$x^2 + y^2 = 16$

Top half of a circle!

8.6 $f(x) = \ln(3-x^2)$

(a) Domain? Can only take logarithm of positive numbers

so $3-x^2 > 0$ so $x^2 < 3$

so $-\sqrt{3} < x < \sqrt{3}$.

x-intercepts? $y=0 \Rightarrow \ln(3-x^2) = 0$

$$3-x^2 = 1$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Points $(\sqrt{2},0)$ $(-\sqrt{2},0)$

y-intercepts? $x=0 \Rightarrow y = \ln(3-0^2) = \ln 3$

We can just leave it as $\ln 3$.

Point $(0, \ln 3)$.

For graphing purposes, we might want to know than $\ln 3$ is a little more than 1.

(Remember $e \approx 2.7$ is slightly less than 3, so $\ln 3$ is slightly more than $\ln e = 1$)

(b) $f'(x) = \dfrac{1}{3-x^2} \cdot -2x = \dfrac{-2x}{3-x^2}$

Critical points? When is f' zero or undefined?

When top or bottom is zero.

$-2x = 0 \Rightarrow x = 0$

$3 - x^2 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

	$x=-\sqrt{3}$		$x=0$		$x=\sqrt{3}$
$-2x$		$+ + +$	0	$- - -$	
$3-x^2$		$+ + +$		$+ + +$	
		f' positive f increasing		f' negative f decreasing	

$\uparrow$

Local MAX at $x=0$

(c) Since $f'(x) = \dfrac{-2x}{3-x^2}$ or $\dfrac{2x}{x^2-3}$

we have $f''(x) = \dfrac{(2x)'(x^2-3) - 2x(x^2-3)'}{(x^2-3)^2}$

$$f''(x) = \frac{2(x^2-3) - 2x \cdot 2x}{(x^2-3)^2}$$

$$= \frac{2x^2 - 6 - 4x^2}{(x^2-3)^2} = \frac{-2x^2-6}{(x^2-3)^2}$$

$$= \frac{-2(x^2+3)}{(x^2-3)^2}.$$

Squared!

Denominator is never negative.

When might this change sign?

-2 is never 0 (always neg.)

x^2+3 is never 0 (always pos.)

x^2-3 is 0 only if $x = \pm\sqrt{3}$

but those are not in the domain.

So f'' is always negative.

f is always concave down.

How does $f(x) = \ln(3-x^2)$ behave near the ends of the domain?

If $x \to \pm\sqrt{3}$ then $3-x^2 \to 0$ so $\ln(3-x^2) \to -\infty$.

Rough graph: