QUESTION: Find all intervals where $f(x)$ is increasing, decreasing, concave up, or concave down. Also draw the graph of f.

$$f(x) = \frac{x^2}{x^2+3}$$

SOLUTION: We'll need the first and second derivatives:

$$f'(x) = \frac{(x^2)'(x^2+3) - (x^2)(x^2+3)'}{(x^2+3)^2}$$

$$= \frac{2x(x^2+3) - x^2 \cdot 2x}{(x^2+3)^2} = \frac{2x^3+6x-2x^3}{(x^2+3)^2}$$

$$= \frac{6x}{(x^2+3)^2} \quad \leftarrow \text{This is } f'.$$

Then $$f''(x) = \frac{(6x)'(x^2+3)^2 - 6x\underbrace{\left((x^2+3)^2\right)'}_{\text{this part is like } (w^2)' \text{ where } w \text{ is a function}}}{\left((x^2+3)^2\right)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$$

$$f''(x) = \frac{6\cdot(x^2+3)^2 - 6x\cdot \overbrace{2(x^2+3)\cdot 2x}^{\text{this part comes from } (w^2)' = 2w\cdot w' \text{ (chain rule)}}}{(x^2+3)^4}$$

Then what? Expanding the top won't help.
Instead, take out the <u>common factor</u> in the top.

$$f''(x) = \frac{(x^2+3)\left[6\cdot(x^2+3) - 6x\cdot 2\cdot 2x\right]}{(x^2+3)^4}$$

$$= \frac{6\cdot(x^2+3) - 6x\cdot 2\cdot 2x}{(x^2+3)^3} = \frac{6x^2+18-24x^2}{(x^2+3)^3}$$

$$= \frac{18-18x^2}{(x^2+3)^3} \quad \text{or} \quad \frac{18(1-x^2)}{(x^2+3)^3} \quad \text{or} \quad \frac{18(1+x)(1-x)}{(x^2+3)^3}.$$

This is f''.

Next, where is $f'=0$? $\Rightarrow$ where is f' positive or negative?
Where is $f''=0$? $\Rightarrow$ where is f'' positive or negative?
Also, where are f' or f'' UNDEFINED (if anywhere)?

$$f = \frac{x^2}{x^2+3} \qquad f' = \frac{6x}{(x^2+3)^2} \qquad f'' = \frac{18(1-x^2)}{(x^2+3)^3}$$

When <u>might</u> f' change sign? When top <u>or</u> bottom is 0.

Top $= 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

Bottom $= 0 \Rightarrow$ never, because x^2+3 is positive.

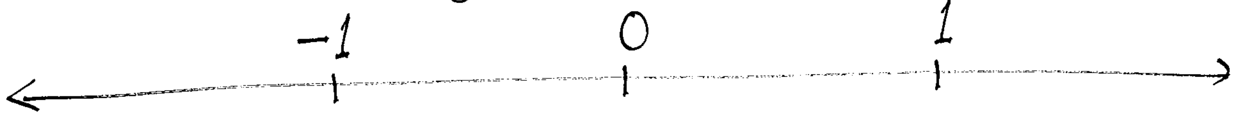
When <u>might</u> f'' change sign? When top <u>or</u> bottom is 0.

Top $= 0 \Rightarrow 1-x^2 = 0 \Rightarrow x = -1$ or $x = 1$

Bottom $= 0 \Rightarrow$ never, because x^2+3 is positive.

So, complete list of x values where f' <u>or</u> f'' <u>might</u> change sign is: $x=-1$, $x=0$, $x=1$.

Number line showing all possible values of x:



f' — — — — — — — 0 + + + + + + + +

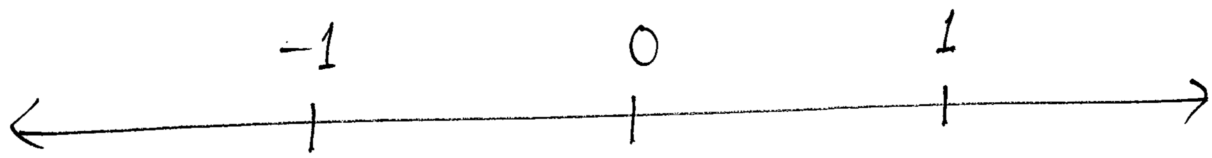
(6 is pos.)
(x^2+3 is pos.)

How do we get this? Either plug in test values, or think holistically.

f'' — — — 0 + + + + + + + + + + 0 — — — —

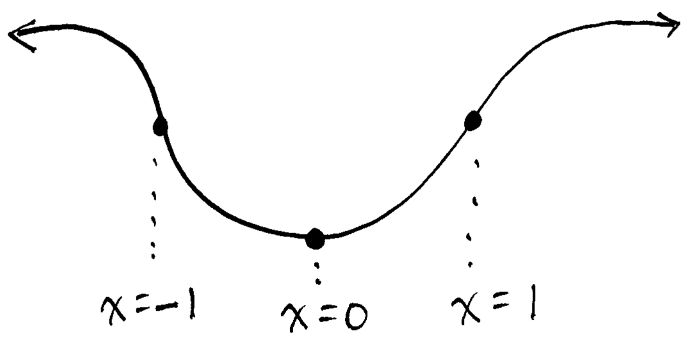
(18 is pos.)
(x^2+3 is pos.)

f' neg	f' neg	f' pos	f' pos
f'' neg	f'' pos	f'' pos	f'' neg



If x is here then $f'<0$, $f''<0$	If x is here then $f'<0$, $f''>0$	If x is here then $f'>0$, $f''>0$	If x is here then $f'>0$, $f''<0$
so f is decr, f is conc down	so f is decr, f is conc up	so f is incr, f is conc up	so f is incr, f is conc down

Overall shape of graph of $f(x) = \dfrac{x^2}{x^2+3}$ is something like:



AND we can also use knowledge of "end behavior" (limits at infinity)

$$\lim_{x \to \pm\infty} \frac{x^2}{x^2+3} = 1 \qquad \frac{1x^2 + \text{smaller powers}}{1x^2 + \text{smaller powers}}$$

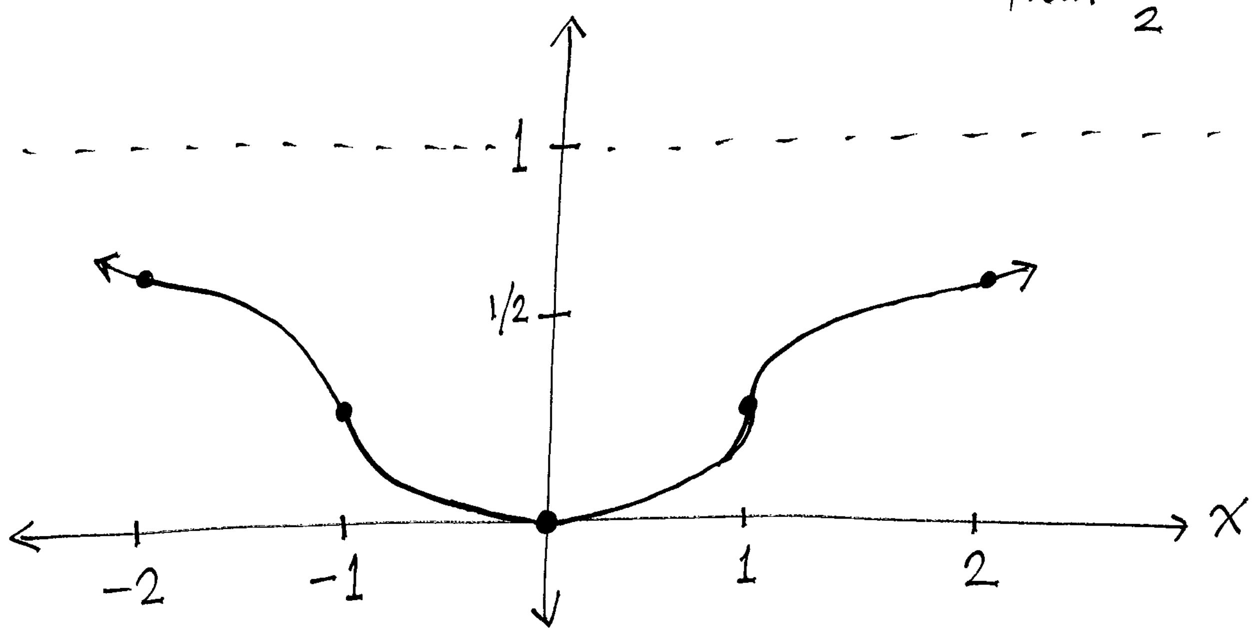
Special points: $f(0) = \dfrac{0^2}{0^2+3} = \dfrac{0}{0+3} = \dfrac{0}{3} = 0$

$$f(\pm 1) = \frac{(\pm 1)^2}{(\pm 1)^2+3} = \frac{1}{1+3} = \frac{1}{4}$$

Maybe a couple more points for very nice picture

$$f(\pm 2) = \frac{(\pm 2)^2}{(\pm 2)^2+3} = \frac{4}{4+3} = \frac{4}{7}$$

which is a little more than $\frac{1}{2}$



f is decreasing and concave down

f decr and conc up

f incr and conc up

f is increasing and concave down

Inflection points at $x=-1$ and $x=1$

Minimum at $x=0$