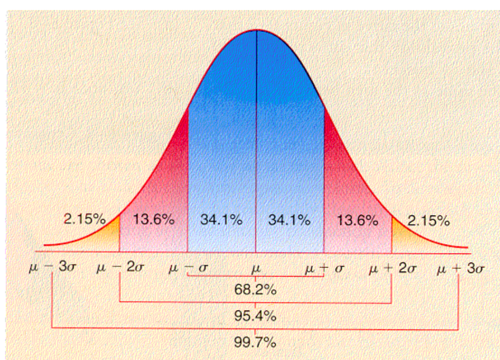
Chapter 0
Basic Prerequisite Knowledge

0.1 Distributions: Normal, Chi-square, *t* and *F*

Normal Distribution: The normal distribution is the most important and most widely used distribution in statistics. A random variable X is said to have a normal distribution with mean μ and standard deviation σ, if its *probability density function* (**pdf**) can be written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty \le x \le \infty$$

Here μ and σ^2 are the mean and variance, respectively, of the distribution. μ also known as location parameter and σ is known as scale parameter. The normal curve is given below.



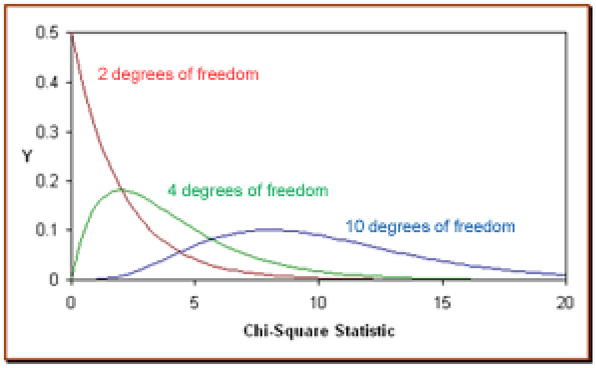
The percentage points for normal distribution are given on page 542.

Chi-square Distribution

A random variable X is said to have a chi-square (χ^2)distribution with mean v degrees of freedom (DF) if its *probability density function* (**pdf**) can be written as

$$f(x) = \frac{e^{-\nu/2}x^{-\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}}, \text{ x} > 0 \text{ and } \nu > 0$$

The shape of the distribution depends on the DF. The mean and variance of X are respectively, E(X)=v and V(X)=2v. As the degrees of freedom increase, the chi-square distribution approaches a normal distribution.



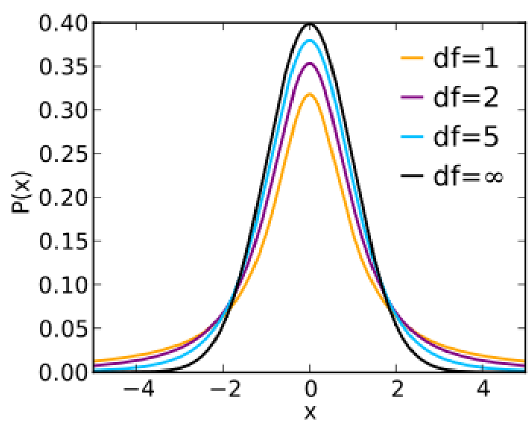
The percentage points for chi-square distribution are given on page 544.

Student's t-Distribution

A random variable X is said to have a Student's *t* distribution with v DF, if its *probability density function* (**pdf**) can be written as

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu\pi)^{1/2}\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

As the degrees of freedom increase, the *t* distribution approaches a normal distribution.

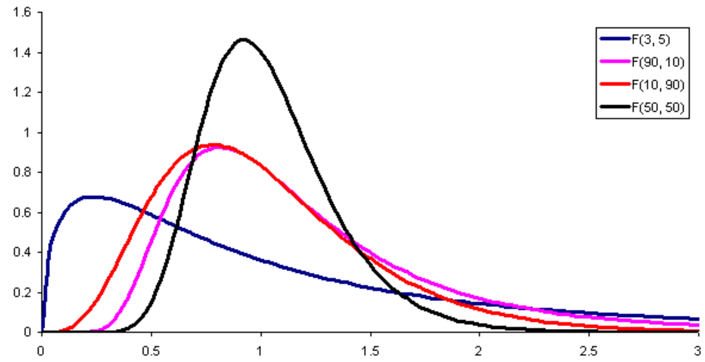


The percentage points for *t* distribution are given on page 545.

F-Distribution

The F-distribution with *m* and *n* degrees of freedom is defined as follows

$$f(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{x^{(m/2-1)}}{\left(1+\frac{m}{n}x\right)^{(m+n)/2}}$$



For *n*=1, F-distribution reduces to *t* distribution with *m* DF. The percentage points for F- distribution is given on page 546.

0.2 Confidence Interval and Test Statistic

Interval Estimator:

The general definition for the confidence interval for mean is

$$\text{Estimator} \pm \text{Margin of Error}$$
$$\text{Estimator} \pm \text{Table Value} \times \text{SE(Estimator)}$$
$$\text{LCL} = \text{Estimator} - \text{Margin of Error}, \ \text{UCL} = \text{Estimator} + \text{Margin of Error}$$

A *1-a level* **confidence interval** for a parameter is an interval computed from sample data by a method that has probability *1-a* of producing an interval containing the true value of the parameter.

Confidence width= UCL – LCL

Confidence Interval for μ (n is large)

A (1-α)100% confidence interval (CI) for μ is

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the *(α/2)th* quantile of a standard normal random variable. This interval is exact when the population distribution is Normal and is approximately correct when *n* is large.

Confidence Interval for μ (n is small)

$$\bar{x} - t_{\alpha/2,v}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2,v}\frac{\sigma}{\sqrt{n}},$$

where $t_{\alpha/2,v}$ is the *(α/2)th* quantile of a *t* random variable with v DF. The normality assumption for the data is required for this interval.

Test of Hypothesis

The Elements of a Test of Hypothesis: The objective of a statistical test is to test a hypothesis concerning the value of one or more population parameters. A statistical test mainly involves the following **five** elements:

1. **Null and alternative hypotheses**:

One sided alternative and two sided alternative

2. **Test statistic:** A statistic computed from sample measurements that will be used to test the null hypothesis. OR, The sample quantity on which the decision to support H_0 or *Ha* is based is called the test statistic.

$$z = \frac{\text{Estimate - Null value}}{\text{SD of the estimate}}$$

3. **P-value:** The P-value is the probability that the test statistic will take on a value that is at least as extreme as the observed value of the test statistic when the null hypothesis is true.

OR

Rejection region (RR): We will reject the null hypothesis at α level of significance if the value of the test statistic falls in the rejection region.

4. Decision Rules: Smaller the P-value, the more likely to reject the null hypothesis. More specifically, reject H_0 at a% level of significance, if

$$P - value \leq a$$

5. Conclusion: Finally write the conclusion about the null hypothesis**.**

Type I and Type II Errors

Two kinds of errors are committed when testing the hypotheses or making decision about the null hypothesis.

Type I Error: If H_0 is rejected when it is true (ie, rejecting a true null hypothesis). That is, a true null hypothesis can be incorrectly rejected.

Type II Error: If H_0 is fail to reject when it is false (ie, accepting a false null hypothesis). That is a false null hypothesis can fail to be rejected.

The goodness of statistical test of hypothesis is measured by the probabilities of making a type I or a type II error, denoted by a and β respectively.

0.3 Elements of Matrix Algebra

Vectors: An array ***x*** of real numbers $x_1, x_2, \ldots, x_n$ is called a vector and it is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad x' = [x_1, x_2, \cdots, x_n]$$

where the prime (') denotes the operation of transposing a column to a row.

For example a'=[1, 6, 3, 2, 1] is a row vector of length 5 and $b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a column vector of length 3.

Matrix:

A matrix is any rectangular array of real numbers. We denote an arbitrary array of *n* rows and *p* columns by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix}$$

We also write $\{a_{ij}\}$ for **A** (bold A) and $A_{n\times p} = \underset{n\times p}{A}$ to indicate the numbers of rows and columns. [here n rows and p columns]. Vectors are matrices with one column.

Note: Usually, bold face upper cases are used for matrices and lower cases are used for vectors.
Equality: Two matrices are equal if and only if their dimensions are identical; and they have exactly the same entries in the same positions.

Transpose of a Matrix: The transpose operation **A′** [or $\mathbf{A}^T$] of a matrix changes the columns into rows, so that the first column of A becomes the first row of **A′**. If **A** is an mxn matrix then **A′** would be an nxm matrix.

(A′)′= A and for two matrices, (AB)′=B′A′ Similarly for three matrices A, B, C, we have (ABC)′=C′B′A′

Symmetric Matrix

Let **A** be a kxk matrix (*square matrix OR a matrix of order k)* is said to be symmetric if it satisfies

$$A' = A \tag{1}$$

where A′ denotes the transpose, so $a_{ij}=a_{ji}$. This also implies

$$A^{-1}A' = I \tag{2}$$

where I is the *identity matrix*. For example,

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 4 \\ 2 & 4 & 7 \end{bmatrix}$$

is a symmetric matrix and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix.

The Matrix Product: The product **AB** of an *mxn* matrix **A**={a_{ij}} and an nxk matrix **B**={b_{ij}} is the mxk **C** whose elements are c_{ij} and defined below

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}; \mathrm{i} = 1,2,..,\mathrm{m} \text{ and } \mathrm{j} = 1,2,..,\mathrm{k}$$

Example: Suppose we have following two matrices and one vector and find **AB** and **CA**.

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 4 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

Inverse Matrix: The inverse $\mathbf{A}^{-1}$ of a square matrix **A** is the unique matrix such that

$$\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}=\mathbf{A}\mathbf{A}^{-1}.$$

Orthogonal Matrix: A matrix **A** will be called an orthogonal matrix *iff,*

$$A^{-1}=A'$$

For an orthogonal matrix, $A'A=AA'=I$ (identity matrix)

Determinant of a Matrix: The determinant of the square kxk matrix $\mathbf{A}=\{a_{ij}\}$. Denoted by $|\mathbf{A}|$, is the scalar

$$|A|=a_{11} \qquad \text{if } k=1$$

$$|A|=\sum_{j=1}^{k} a_{1j}|A_{1j}|(-1)^{1+j} \qquad \text{if } k>1$$

where A_{1j} is the *(k-1)x(k-1)* matrix obtained by deleting the first row and *jth* column of A.

Example: Find the determinant of the following matrix.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Singular Matrix: A square matrix which does not have an inverse. A matrix is singular if and only if its determinant is zero.