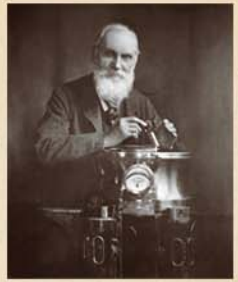
Rembrandt's "Dido Divides the Oxhide" (mid-1600s)



ISOPERIMETRICAL PROBLEMS.

[*Being a Friday evening Lecture delivered to the Royal Institution, May 12th*, 1893.]

Dido, B.C. 800 or 900.
Horatius Cocles, B.C. 508.
Pappus, Book V., A.D. 390.
John Bernoulli, A.D. 1700.
Euler, A.D. 1744.
Maupertuis (Least Action), b. 1698, d. 1759.
Lagrange (Calculus of Variations), 1759.
Hamilton (Actional Equations of Dynamics), 1834.
Liouville, 1840 to 1860.

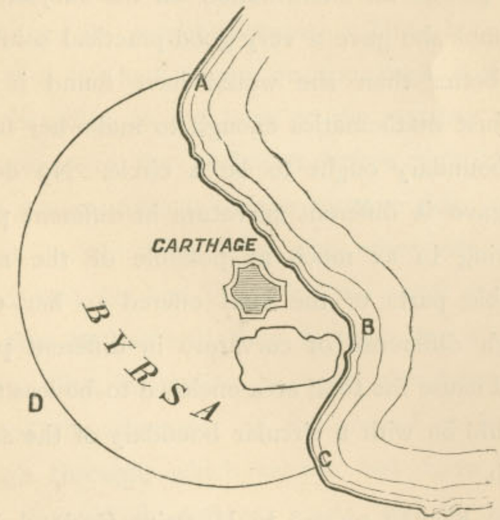
THE first isoperimetrical problem known in history was practically solved by Dido, a clever Phœnician princess, who left her Tyrian home and emigrated to North Africa, with all her property and a large retinue, because her brother Pygmalion murdered her rich uncle and husband Acerbas, and plotted to defraud her of the money which he left. On landing in a bay about the middle of the north coast of Africa she obtained a grant from Hiarbas, the native chief of the district, of as much land as she could enclose with an ox-hide. She cut the ox-hide into an exceedingly long strip, and succeeded in enclosing between it and the sea a very valuable territory [1] on which she built Carthage.

The next isoperimetrical problem on record was three or four hundred years later, when Horatius Cocles, after saving his country by defending the bridge until it was destroyed by the Romans behind him, saved his own life and got back into Rome by swimming the Tiber under the broken bridge, and was rewarded by his grateful countrymen with a grant of as much land as he could plough round in a day.

In Dido's problem the greatest value of land was to be enclosed by a line of given length.

[1] Called Byrsa, from βύρσα, the hide of a bull. [Smith's *Dictionary of Greek and Roman Biography and Mythology*, article "Dido."]

If the land is all of equal value the general solution of the problem shows that her line of ox-hide should be laid down in a circle. It shows also that if the sea is to be part of the boundary, starting, let us say, southward from any



given point, A, of the coast, the inland bounding line must at its far end cut the coast line perpendicularly. Here, then, to complete our solution, we have a very curious and interesting, but not at all easy, geometrical question to answer:—What must be the radius of a circular arc, A D C, of given length, and in what direction must it leave the point A, in order that it may cut a given curve, A B C, perpendicularly at some unknown point, C? I don't believe Dido could have passed an examination on the subject, but no doubt she gave a very good practical solution, and better than she would have found if she had just mathematics enough to make her fancy the boundary ought to be a circle. No doubt she gave it different curvature in different parts to bring in as much as possible of the more valuable parts of the land offered to her, even though difference of curvature in different parts would cause the total area enclosed to be less than it would be with a circular boundary of the same length.

The Roman reward to Horatius Cocles brings in quite a new idea, now well known in the general subject of isoperimetrics: the greater or less speed attainable according to the nature of the country through which the line travelled over passes. If it had been equally easy to

plough the furrow in all parts of the area offered for enclosure, and if the value of the land per acre was equal throughout, Cocles would certainly have ploughed as nearly in a circle as he could, and would only have deviated from a single circular path if he found that he had misjudged its proper curvature. Thus, he might find that he had begun on too large a circle and, in order to get back to the starting point and complete the enclosure before nightfall, he must deviate from it on the concave side; or he would deviate from it on the other side if he found that he had begun on too small a circle and that he had still time to spare for a wider sweep. But, in reality, he must also have considered the character of the ground he had to plough through, which cannot but have been very unequal in different parts, and he would naturally vary the curvature of his path to avoid places where his ploughing must be very slow, and to choose those where it would be most rapid.

He must also have had, as Dido had, to consider the different value of the land in different parts, and thus he had a very complex problem to practically solve. He had to be guided both by the value of the land to be enclosed and the speed at which he could plough according to the path chosen; and he had a very brain-trying task to judge what line he must follow to get the largest value of land enclosed before night.

These two very ancient stories, whether severe critics will call them mythical or allow them to be historic, are nevertheless full of scientific interest. Each of them expresses a perfectly definite case of the great isoperimetrical problem to which the whole of dynamics is reduced by the modern mathematical methods of Euler, Lagrange, Hamilton, and Liouville (Liouville's Journal, 1840–1850). In Dido's and Horatius Cocles' problems, we find perfect illustrations of all the fundamental principles and details of the generalised treatment of dynamics which we have learned from these great mathematicians of the eighteenth and nineteenth centuries.

Nine hundred years after the time of Horatius

Cocles we find, in the fifth Book of the collected Mathematical and Physical Papers of Pappus of Alexandria, still another idea belonging to isoperimetrics—the economy of valuable material used for building a wall; which, however, is virtually the same as the time per yard of furrow in Cocles' ploughing. In this new case the economist is not a clever princess, nor a patriot soldier; but a humble bee who is praised in the introduction to the book not only for his admirable obedience to the Authorities of his Republic, for the neat and tidy manner in which he collects honey, and for his prudent thoughtfulness in arranging for its storage and preservation for future use, but also for his knowledge of the geometrical truth that a "hexagon can enclose more honey than a square or a triangle with equal quantities of building material in the walls," and for his choosing on this account the hexagonal form for his cells. Pappus, concluding his introduction with the remark that bees only know as much of geometry as is practically useful to them, proceeds to apply what he calls his own superior human intelligence to investigation of useless knowledge, and gives results in his Book V., which consists of fifty-five theorems and fifty-seven propositions on the areas of various plane figures having equal circumferences. In this Book, written originally in Greek, we find (Theorem IX. Proposition X.) the expression "isoperimetrical figures," which is, so far as I know, the first use of the adjective "isoperimetrical" in geometry; and we may, I believe, justly regard Pappus as the originator, for mathematics, of *isoperimetrical problems*, the designation technically given in the nineteenth century[1] to that large province of mathematical and engineering science in which different figures having equal circumferences, or different paths between two given points, or between some two points on two given curves, or on one given curve, are compared in connection with definite questions of greatest efficiency and smallest cost.

[1] Example, Woodhouse's *Isoperimetrical Problems*, Cambridge, 1810.

In the modern engineering of railways an isoperimetrical problem of continual recurrence is the laying out of a line between two towns along which a railway may be made at the smallest prime cost. If this were to be done irrespectively of all other considerations, the requisite datum for its solution would be simply the cost per yard of making the railway in any part of the country between the two towns. Practically the solution would be found in the engineers' drawing office by laying down two or three trial lines to begin with, and calculating the cost of each, and choosing the one of which the cost is least. In practice various other considerations than very slight differences in the cost of construction will decide the ultimate choice of the exact line to be taken, but if the problem were put before a capable engineer to find very exactly the line of minimum total cost, with an absolutely definite statement of the cost per yard in every part of the country, he or his draughtsmen would know perfectly how to find the solution. Having found some-thing near the true line by a few rough trials they would try small deviations from the rough approximation, and calculate differences of cost for different lines differing very little from one another. From their drawings and calculations they would judge by eye which way they must deviate from the best line already found, to find one still better. At last they would find two lines for which their calculation shows no difference of cost. Either of these might be chosen; or, according to judgment, a line mid-way between them, or somewhere between them, or even not between them but near to one of them, might be chosen, as the best approximation to the exact solution of the mathematical problem which they care to take the labour of trying for. But it is clear that if the price per yard of the line were accurately given (however determined or assumed) there would be an absolutely definite solution of the problem, and we can easily understand that the skill available in a good engineer's drawing-office would suffice to find the solution with any degree of accuracy

that might be prescribed; the minuter the accuracy to be attained the greater the labour, of course. You must not imagine that I suggest, as a thing of practical engineering, the attainment of minute accuracy in the solution of a problem thus arbitrarily proposed; but it is interesting to know that there is no limit to the accuracy to which this ideal problem may be worked out by the methods which are actually used every day by engineers in their calculations and drawings.

The modern method of the "calculus of variations," brought into the perfect and beautiful analytical form in which we now have it by Lagrange, gives for this particular problem a theorem which would be very valuable to the draughtsman if he were required to produce an exceedingly accurate drawing of the required curve. The curvature of the curve at any point is convex towards the side on which the price per unit length of line is less, and is numerically equal to the rate per mile perpendicular to the line at which the Neperian logarithm of the price per unit length of the line varies. This statement would give the radius of curvature in fraction of a mile. If we wish to have it in yards we must take the rate per yard at which the Neperian logarithm of the price per unit length of the line varies. I commend the Neperian logarithm of price in pounds, shillings and pence, to our Honorary Secretary, to whom no doubt it will present a perfectly clear idea; but less powerful men would prefer to reckon the price in pence, or in pounds and decimals of a pound. In every possible case of its subject the "calculus of variations" gives a theorem of curvature less simple in all other cases than in that very simple case of the railway line of minimum first cost, but always interpretable and intelligible according to the same principles.

Thus in Dido's problem we find by the calculus of variations that the curvature of the enclosing line varies in simple proportion to the value of the land at the places through which it passes; and the curvature at any one place is determined by the condition that the whole length of the ox-hide just completes the enclosure.

The problem of Horatius Cocles combines the railway problem with that of Dido. In it the curvature of the boundary is the sum of two parts; one, as in the railway, equal to the rate of variation perpendicular to the line, of the Neperian logarithm of the cost in time per yard of the furrow (instead of cost in money per yard of the railway); the other varying proportionally to the value of the land as in Dido's problem, but now divided by the cost per yard of the line which is constant in Dido's case. The first of these parts, added to the ratio of the money-value per square yard of the land to the money-cost per lineal yard of the boundary (a wall, suppose), is the curvature of the boundary when the problem is simply to make the most you can of a grant of as much land as you please to take provided you build a proper and sufficient stone wall round it at your own expense. This problem, unless wall-building is so costly that no part of the offered land will pay for the wall round it, has clearly a determinate finite solution if the offered land is an oasis surrounded by valueless desert.

It has also a determinate finite solution even though the land be nowhere valueless, if the wall is sufficiently more and more expensive at greater and greater distances from some place where there are quarries, or habitations for the builders.

The simplified case of this problem, in which all equal areas of the land are equally valuable, is identical with the old well-known Cambridge dynamical plane problem of finding the motion of a particle relatively to a line of reference revolving uniformly in a plane: to which belongs that considerable part of the "Lunar Theory" in which any possible motion of the moon is calculated on the supposition that the centre of gravity of the earth and moon moves uniformly in a circle round the sun, and that the motions of the earth and moon are exactly in this plane. The rule for curvature which I have given you expresses in words the essence of the calculation, and suggests a graphic method for finding solutions by which not uninteresting approximations[1]

[1] Kelvin, "On graphic solution of dynamical problems." *Phil. Mag.* 1892 (2nd half-year).

to the cusped and looped orbits of G. F. Hill[1] and Poincare[2] can be obtained without disproportionately great labour.

In the dynamical problem, the angular velocity of the revolving line of reference is numerically equal to half the value of the land per square yard; and the relative velocity of the moving particle is numerically equal to the cost of the wall per lineal yard in the land question.

But now as to the proper theorem of curvature for each case; both Dido and Horatius Cocles no doubt felt it instinctively and were guided by it, though they could not put it into words, still less prove it by the "calculus of variations." It was useless knowledge to the bees, and, therefore, they did not know it; because they had only to do with straight lines. But as you are not bees I advise you all, even though you have no interest in acquiring as much property as you can enclose by a wall of given length, to try Dido's problem

[1] Hill, *Researches in the Lunar Theory*, Part 3. National Academy of Sciences, 1887.

[2] *Méthodes Nouvelles de la Mécanique Céleste*, p. 109 (1892).

for yourselves, simplifying it, however, by doing away with the rugged coast line for part of your boundary, and completing the enclosure by the wall itself. Take forty inches of thin soft black thread with its ends knotted together and let it represent the wall ; lay it down on a large sheet of white paper and try to enclose the greatest area with it you can. You will feel that you must stretch it in a circle to do this, and then, perhaps, you will like to read Pappus (Liber V. Theorema II. Propositio II.) to find mathematical demonstration that you have judged rightly for the case of all equal areas of the enclosed land equally valuable. Next try a case in which the land is of different value in different parts. Take a square foot of white paper and divide it into 144 square inches to represent square miles, your forty inches of endless thread representing a forty miles wall to enclose the area you are to acquire. Write on each square the value of that particular square mile of land, and place your endless thread upon the paper, stretched round a large number of smooth pins stuck through the paper into a

drawing-board below it, so as to enclose as much value as you can, judging first roughly by eye and then correcting according to the sum of the values of complete squares and proportional values of parts of squares enclosed by it. In a very short time you will find with practical accuracy the proper shape of the wall to enclose the greatest value of the land that can be enclosed by forty miles of wall. When you have done this you will understand exactly the subject of the calculus of variations, and those of you who are mathematical students may be inclined to read Lagrange, Woodhouse, and other modern writers on the subject. The problem of Horatius Cocles, when not only the different values of the land in different places but also the different speed of the plough according to the nature of the ground through which the furrow is cut are taken into consideration, though more complex and difficult, is still quite practicable by the ordinary graphic method of trial and error. The analytical method of the calculus of variations, of which I have told you the result, gives simply the proper curvature for the furrow in any particular direction through any particular place. It gives this and it cannot give anything but this, for any plane isoperimetrical problem whatever, or for any isoperimetrical problem on a given curved surface of any kind.

Beautiful, simple, and clear as isoperimetrics is in geometry, its greatest interest, to my mind, is in its dynamical applications. The great theorem of least action, somewhat mystically and vaguely propounded by Maupertuis, was magnificently developed by Lagrange and Hamilton, and by them demonstrated to be not only true throughout the whole material world, but also a sufficient foundation for the whole of dynamical science.

It would require nearly another hour if I were to explain to you fully this grand generalisation for any number of bodies moving freely, such as the planets and satellites of the solar system, or any number of bodies connected by cords, links, or mutual pressures between hard surfaces, as in a spinning-wheel, or lathe and treadle, or a steam-engine, or a crane, or a machine of any kind; but even if it were convenient to you to remain here an

hour longer, I fear that two hours of pure mathematics and dynamics might be too fatiguing. I must, therefore, perforce limit myself to the two-dimensional, but otherwise wholly comprehensive, problems of Dido and Horatius Cocles. Going back to the simpler included case of the railway of minimum cost between two towns, the dynamical analogue is this:—For price per unit length of the line substitute the velocity of a point moving in a plane under the influence of a given conservative system of forces, that is to say, such a system that when material particles not mutually influencing one another are projected from one and the same point in different directions, but with equal velocities, the subsequent velocity of each is calculable from its position at any instant, and all have equal velocities in travelling through the same place whatever may be their directions. The theorem of curvature, of which I told you in connection with the railway engineering problem, is now simply the well-known elementary law of relation between curvature and centrifugal force of the motion of a particle.

The motion of a particle in a plane is, as Liouville has proved, a case to which every possible problem of dynamics involving just two freedoms to move can be reduced. But to bring you to see clearly its relation to isoperimetrics, I must tell you of another admirable theorem of Liouville's, reducing to a still simpler case the most general dynamics of two-freedoms motion. Though not all mathematical experts, I am sure you can all perfectly understand the simplicity of the problem of drawing the shortest line on any given convex surface, such as the surface of this block of wood (shaped to illustrate Newton's dynamical theory of the elliptic motion of a planet round the sun) which you see on the table before you. I solve the problem practically by stretching a thin cord between the two points, and pressing it a little this way or that way with my fingers till I see and feel that it lies along the shortest distance between them. And now, when I tell you that Liouville has reduced to this splendidly simple problem of drawing a shortest line (geodetic line it is called) on any given curved surface every conceivable

problem of dynamics involving only two freedoms to move, I am sure you will understand sufficiently to admire the great beauty of this theorem.

The doctrine of isoperimetrical problems in its relation to dynamics is very valuable in helping to theoretical investigation of an exceedingly important subject for astronomy and physics—the stability of motion, regarding which, however, I can only this evening venture to show you some experimental illustrations.

The lecture was concluded with experiments illustrating—

1. Rigid bodies (teetotums, boys' tops, ovals, oblates, &c.) placed on a horizontal plane, and caused to spin round on a vertical axis, and found to be thus rendered stable or unstable according as the equilibrium without spinning is unstable or stable.

2. The stability or instability of a simple pendulum whose point of support is caused to vibrate up and down in a vertical line, investigated mathematically by Lord Rayleigh.

3. The crispations of a liquid supported on a

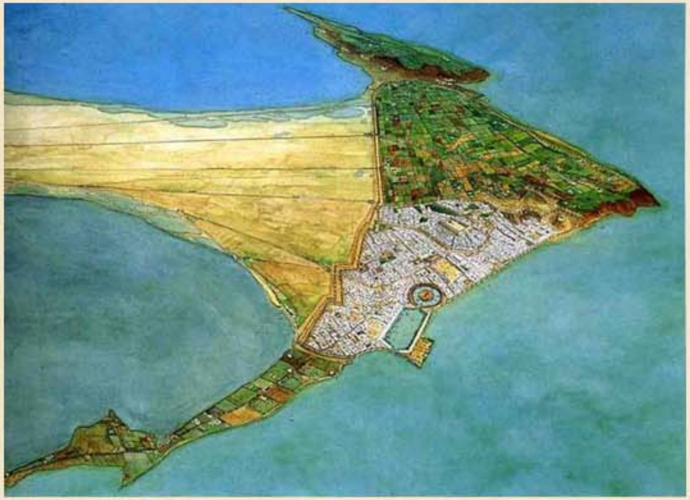
vibrating plate, investigated experimentally by Faraday; and the instability of a liquid in a glass jar, vibrating up and down in a vertical line, demonstrated mathematically by Lord Rayleigh.

4. The instability of water in a prolate hollow vessel, and its stability in an oblate hollow vessel, each caused to rotate rapidly round its axis of figure,[1] which were announced to Section A of the British Association at its Glasgow meeting in 1876 as results of an investigation not then published, and which has not been published up to the present time.

[1] *Nature*, 1877, vol. 15, p. 297, "On the Precessional Motion of a Liquid."

"Dido cuts the Oxhide" by Johann Bocksberger (1564)
Façade Painting, Regensburg Rathaus (Townhall), Germany

Carthage according to the Nuremberg Chronicle, Folios XLv-XLIr
The 3rd Age from the Birth of Abraham to the Kingdom of David
(AD 1493)



Punic Carthage from Goletta to Gammarth: Three cities in one, according to Sir Grenville Temple (1835): Cothon (or Katum, i.e., the Harbor), Bosra (Byrsa) and Magar (or Magaria, i.e, la Marsa)

Anonymous Venitian (16th century): Dido Founding Carthage.
Villa Giusti, Magnadola de Cessalto

Francesco Fontebasso's "Dido cuts the Oxhide" at the Albertina, Vienna

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