On some spectral inequalities for non-elliptic partial differential operators

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Carthage, Tunisia, May 24-29, 2010

Plan of the talk:

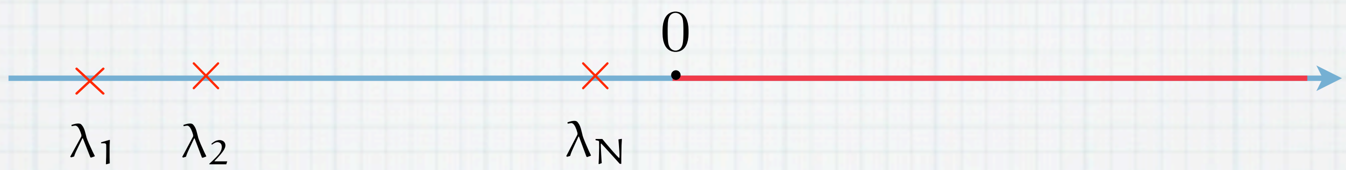
- Spectral inequalities for a class non-elliptic operators

(with Fabian Portmann)

- Spectral and Hardy inequalities for Heisenberg Laplacians

I. Let $H = -\Delta - V$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$ and let V be real, $V \to 0$, as $x \to \infty$

Spectrum:



Lieb-Thirring inequalities:

$$\sum_j |\lambda_j|^\gamma \le \frac{C_{\gamma,d}}{(2\pi)^d} \int\int \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, dx d\xi = L_{\gamma,d} \int V(x)_+^{\gamma+d/2} \, dx.$$

This inequality holds true for $d = 1$, $\gamma \ge 1/2$; $d = 2$, $\gamma > 0$; $d \ge 3$, $\gamma \ge 0$.

If $\gamma = 1$ then L-Th inequalities are equivalent to the following generalised Sobolev inequality for an orthonormal system of functions $\{\varphi_k\}_{k=1}^N$ in $L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} [\rho_N(x)]^{(2+d)/d}\, dx \le C_d \sum_{k=1}^N \int_{\mathbb{R}^d} |\nabla \varphi_k(x)|^2\, dx,$$

where

$$\rho_N(x) = \sum_{k=1}^N |\varphi_k(x)|^2.$$

Using the Fourier transform the latter inequality can be rewritten as

$$\int_{\mathbb{R}^d} [\rho_N(x)]^{(2+d)/d}\, dx \le C_d (2\pi)^d \sum_{k=1}^N \int_{\mathbb{R}^d} |\xi|^2 |\hat{\varphi}_k(\xi)|^2\, d\xi, \quad x, \xi \in \mathbb{R}^d.$$

Recently D.S. Barsegyan has obtained L-Th type inequalities in $\mathbb{R}^2$, where the Laplace operator (whose symbol equals $|\xi|^2$) has been substituted by the product $|D_x D_y|$, $D_x = -i\partial/\partial x$. In this case the latter inequality takes the form

$$\int_{\mathbb{R}^2} [\rho_N(x,y)]^2 \, dxdy \leq C\,(\log N + 1) \sum_{k=1}^{N} \int_{\mathbb{R}^2} |\xi\eta||\hat{\varphi}_k(\xi,\eta)|^2 \, d\xi d\eta, \qquad (x,y), (\xi,\eta) \in \mathbb{R}^2,$$

where the constant C is independent of N.
This inequality could be rewritten as an inequality for the negative eigenvalues $\{-\lambda_k\}$ of the operator

$$|D_x||D_y| - V$$

acting in $L^2(\mathbb{R}^2)$.

Let $-\lambda_1 \le -\lambda_2 \le \dots \le -\lambda_N \le \dots$ be the sequence of negative eigenvalues, then for any N,

$$\sum_{k=1}^{N} \lambda_k \le C(\log N + 1) \int_{\mathbb{R}^2} V_+^2(x,y)\, dxdy.$$

Indeed, if $\{\varphi_k\}$ is an orthonormal system of eigenfunctions of the operator $|D_x||D_y| - V$, then by the Cauchy-Schwarz inequality we have

$$-\sum_{k=1}^{N} \lambda_k = \int_{\mathbb{R}^2} |\xi\eta| \sum_{k=1}^{N} |\hat{\varphi}_k(\xi,\eta)|^2\, d\xi d\eta - \int_{\mathbb{R}^2} V \sum_{k=1}^{N} |\varphi_k(x,y)|^2\, dxdy$$

$$\ge C(\log N+1)^{-1} \int_{\mathbb{R}^2} [\rho_N(x,y)]^2\, dxdy - \Big(\int_{\mathbb{R}^2} V^2\, dxdy\Big)^{1/2} \Big(\int_{\mathbb{R}^2} [\rho_N(x,y)]^2\, dxdy\Big)^{1/2}.$$

Minimizing the right hand side with respect to

$$X = \Big(\int_{\mathbb{R}^2} [\rho_N(x,y)]^2\, dxdy\Big)^{1/2}$$

we complete the proof.

Remark

The generalised Sobolev inequality and also

$$\sum_{k=1}^{N} \lambda_k \le C(\log N + 1) \int_{\mathbb{R}^2} V_+^2(x,y)\, dxdy.$$

are sharp.
However, it does not give a satisfactory inequality for the sum of all negative eigenvalues, because its right hand side depends on $\log N + 1$. When $d = 2$, estimates for the number of negative eigenvalues even for Schrödinger operators is a delicate problem. Necessary and sufficient condition for the finiteness of the negative spectrum is so far not known.

Let us consider

$$D_x^2 D_y^2 u - Vu = -\lambda u, \quad u(x,0) = u(0,y) = 0.$$

in $L^2(\mathbb{R}^2_{++})$, where $\mathbb{R}^2_{++} = (0,\infty)^2$.

The following theorem is our main result.

Theorem

Let $\gamma \geq 1/2$. Then for the negative eigenvalues $\{-\lambda_k\}$ of the operator defined above we have

$$\sum_k \lambda_k^\gamma \leq \frac{1}{4^\gamma} (2\pi)^{-2} (L_{\gamma,1})^2 \mathcal{B}(1/2, \gamma+1) \int_{\mathbb{R}^2_{++}} V_+^{1/2+\gamma} \log(1 + 4xy\sqrt{V_+})\, dxdy,$$

where $L_{\gamma,1}$ are the L-Th constants defined before and

$$\mathcal{B}(a,b) = \int_0^1 t^{a-1}(1-t)_+^{b-1}$$

is the classical Beta-function.

Remark

Differential operator $D_x^2 D_y^2$ is highly non-elliptic and a phase type spectral inequalities are impossible. Some examples of operators with infinite classical phase volume were previously considered in papers of B. Simon, M. Solomyak and M. Solomyak and I. Vulis.

The sharpness of our result could be confirmed by the following argument. Let us consider the operator H in $L^2(\mathbb{R}^2)$

$$Hu(x,y) = (D_x^2 D_y^2 + x^2 + y^2)u(x,y) = \lambda u(x,y).$$

Then it is known (Simon) that for the counting function of its spectrum

$$N(H,\lambda) = \#\{k : \lambda_k < \lambda\}$$

we have the following asymptotic formula

$$N(\lambda) = \pi^{-1}\lambda^{3/2}\log\lambda + o(\lambda^{3/2}\log\lambda), \quad \lambda \to \infty.$$

This formula immediately implies that

$$\sum_k (\lambda - \lambda_k)_+^\gamma = \frac{1}{(\gamma+3/2)\pi}\lambda^{\gamma+3/2}\log\lambda + o(\lambda^{\gamma+3/2}\log\lambda), \quad \lambda \to \infty.$$

We now reduce this problem to the study of the negative spectrum of the operator $D_x^2 D_y^2 - (\lambda - x^2 - y^2)_+$. By our Theorem for $\gamma \geq 1/2$

$$\sum_k (\lambda - \lambda_k)_+^\gamma \leq \frac{1}{4^\gamma}\,(2\pi)^{-2}(L_{\gamma,1})^2\,\mathcal{B}(1/2, \gamma+1)\times$$

$$\times \int_{\mathbb{R}^2_{++}} (\lambda - x^2 - y^2)_+^{1/2+\gamma} \log(1 + 4xy\sqrt{(\lambda - x^2 - y^2)_+})\,dxdy$$

$$\leq C\lambda^{\gamma+3/2}\big(1 + \log(\lambda+1)\big),$$

where C is independent of λ. Applying the Legendre transform we find that for the sum of eigenvalues of the operator $D_x^2 D_y^2 + x^2 + y^2$ we have

$$\sum_{k=1}^N \lambda_k \geq C\frac{N^{5/3}}{1 + \log^{2/3} N}.$$

The latter formula has been also obtained by D.S. Barsegyan.

Proof.

$$\operatorname{Tr}\left((D_x^2 D_y^2)^{\mathcal{D}} - V\right)_-^\gamma \leq \operatorname{Tr}\left(\left(\frac{1}{2}(D_x^2)^{\mathcal{D}} + \frac{1}{8x^2}\right)(D_y^2)^{\mathcal{D}} - V\right)_-^\gamma$$
$$= \operatorname{Tr}\left(\left(\frac{1}{2}(D_x^2)^{\mathcal{D}} + \frac{1}{8x^2}\right)\eta^2 - \hat{\mathbf{V}}\right)_-^\gamma,$$

where $\hat{\mathbf{V}}$ is the integral operator defined by

$$\hat{\mathbf{V}}u(x,\eta) = \int_{\mathbb{R}} \hat{V}(x,\eta-\theta)u(x,\theta)\,d\theta.$$

If $\gamma \geq 1/2$, then can use the L-Th inequality with the operator-valued potential

$$\frac{1}{8x^2}\,\eta^2 - \hat{\mathbf{V}}$$

$$\mathrm{Tr}\,((D_x^2 D_y^2)^{\mathcal{D}} - V)_-^\gamma \leq (2\pi)^{-1} L_{\gamma,1} \int_{\mathbb{R}} \int_0^\infty \mathrm{Tr}\,\Big(\Big(\frac{1}{2}\xi^2 + \frac{1}{8x^2}\Big)\eta^2 - \hat{\mathbf{V}}\Big)_-^\gamma \, dx d\xi$$
$$= (2\pi)^{-1} L_{\gamma,1} \int_{\mathbb{R}} \int_0^\infty \mathrm{Tr}\,\Big(\Big(\frac{1}{2}\xi^2 + \frac{1}{8x^2}\Big)(D_y^2)^{\mathcal{D}} - V\Big)_-^\gamma \, dx d\xi.$$

Using the same argument with respect to y-variable, we find

$$\mathrm{Tr}\,((D_x^2 D_y^2)^{\mathcal{D}} - V)_-^\gamma$$
$$\leq (2\pi)^{-1} L_{\gamma,1} \int_{\mathbb{R}} \int_0^\infty \mathrm{Tr}\,\Big(\Big(\frac{1}{2}\xi^2 + \frac{1}{8x^2}\Big)\Big(\frac{1}{2}(D_y^2)^{\mathcal{D}} + \frac{1}{8y^2}\Big) - V\Big)_-^\gamma \, dx d\xi$$
$$\leq \frac{1}{4^\gamma}(2\pi)^{-2}(L_{\gamma,1})^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_{++}} \Big(\Big(\xi^2 + \frac{1}{4x^2}\Big)\Big(\eta^2 + \frac{1}{4y^2}\Big) - V\Big)_-^\gamma \, dx dy d\xi d\eta.$$

The proof is complete.

Let us consider the self-adjoint operator in $L^2(\mathbb{R}^2)$

$$H_0 = -\Delta + x^2y^2 + x\sigma_1 - y\sigma_2,$$

where $\sigma_{1,2,3}$ are standard Pauli matrices.
It has been proved by G.M.Graf, D.Hasler and J.Hoppe that H_0 could be formally factorized $H_0 = Q^2 \geq 0$, where

$$Q = -i(\partial_x\sigma_1 + \partial_y\sigma_2) + xy\,\sigma_3.$$

Moreover, H_0 acting in $L^2(\mathbb{R}^2)$, has purely absolute continuous spectrum $\Sigma(H) = [0, \infty)$.

Open problem.

Let $V \geq 0$ be a positive function from a suitable functional class.
Find Lieb-Thirring inequalities for the negative eigenvalues of the operator

$$H = H_0 - V.$$

Remark. Some attempt was recently made by Douglas Lundholm.

II. Spectral and Hardy inequalities for Heisenberg Laplacians.

Let X_1, X_2 and X_3 be vector fields in $\mathbb{R}^3$, $x = (x_1, x_2, x_3)$,

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3}, \qquad X_2 = \partial_{x_2} - 2x_1\partial_{x_3}, \qquad X_3 = \partial_{x_3}.$$

$\{X_j\}_{j=1}^3$ are infinitesimal generators of the Heisenberg group which could be described as the set $\mathbb{R}^2 \times \mathbb{R}$ equipped with the group law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(y_2x_1 - y_1x_2)).$$

These vector fields satisfy the canonical commutation relations

$$[X_1, X_2] = -4X_3, \qquad [X_1, X_3] = [X_2, X_3] = 0.$$

We denote by H_0 the second order hypo-elliptic operator

$$H_0 = X_1^*X_1 + X_2^*X_2 = -X_1^2 - X_2^2.$$

Let $z = (x_1, x_2)$ and let us define the distance function

$$d(z, x_3) = (|z|^4 + x_3^2)^{1/4} = ((x_1^2 + x_2^2)^2 + x_3^2)^{1/4}.$$

Lemma. (Hardy Inequality). For any $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$ we have

$$\int (|X_1 u|^2 + |X_2 u|^2)\, dx \geq \int \frac{|z|^2}{d^2} |u|^2\, dx.$$

Proof.

$$\left(X_1 + \frac{X_1 d}{d}\right)^* \left(X_1 + \frac{X_1 d}{d}\right) + \left(X_2 + \frac{X_2 d}{d}\right)^* \left(X_2 + \frac{X_2 d}{d}\right)$$
$$= -X_1^2 - X_2^2 - \frac{|z|^2}{d^2} = H_0 - \frac{|z|^2}{d^2} \geq 0.$$

The operator H_0 could be rewritten as

$$H_0 = -\Delta_z - 4|z|^2\partial_{x_3}^2 - 4\partial_{x_3} T,$$

where

$$T = x_2\partial_{x_1} - x_1\partial_{x_2}.$$

In particular, this immediately implies that if $u = u(|z|, x_3)$ then

$$H_0 u = (-\Delta_z - 4|z|^2\partial_{x_3}^2)u.$$

So the restriction of H_0 on functions depending on $(|z|, x_3)$ coincides with the Grushin operator. Assuming that the Hardy inequality is achieved on "radial" functions its proof could be given by using the "Grushin" vector field

$$G = (\partial_{x_1}, \partial_{x_2}, 2|z|\partial_{x_3}).$$

Namely,

$$\left(G + \frac{Gd}{d}\right)^* \cdot \left(G + \frac{Gd}{d}\right) = (-\Delta_z - 4|z|^2\partial_{x_3}^2) - \frac{|z|^2}{d^2} \geq 0.$$

Remark.

If u is an antisymmetric function w.r.t. (x_1, x_2) $(u(x_1, x_2 = -u(x_2, x_1))$ then due to the extra "contribution" $1/|z|^2$ from $-\Delta_z$ provided by

$$(-\Delta_z - 4|z|^2 \partial_{x_3}^2)$$

we immediately obtain inequalities:

$$\int (|X_1 u|^2 + |X_2 u|^2)\, dx \geq \int \frac{2|z|^2}{d^2} |u|^2\, dx$$

and

$$\int (|X_1 u|^2 + |X_2 u|^2)\, dx \geq \int \frac{|x_3|}{d^2} |u|^2\, dx.$$

Lieb-Thirring inequalities for Heisenberg Laplacians.

Let us consider the operator

$$H_0 - V = -X_1^2 - X_2^2 - V(x).$$

Then for the negative eigenvalues $\{\lambda_k\}$ of this operator we have

$$\sum_k |\lambda_k(H_0 - V)|^\gamma \leq C_\gamma \int_{\mathbb{R}^3} V_+^{\gamma+2}(x)\, dx,$$

for any $\gamma \geq 0$.

Remark.

Sharp constants for these inequalities are unknown.

Open Problem.

Is it true that there is a constant $C = C(\gamma)$ such that

$$\sum_k |\lambda_k(H_0 - V)|^\gamma \leq C(\gamma) \int_{\mathbb{R}^3} \Big(V - \frac{|z|^2}{d^2}\Big)_+^{\gamma+2} dx,$$

for any/some $\gamma > 0$.

Thank you
for your attention

- Spectral inequalities for a Dirichlet Laplacian

(with Leander Geisinger and Timo Weidl)

II. Improved Berezin-Li-Yau inequalities (with L.Geisinger and T.Weidl)

Let $\Omega \subset \mathbb{R}^d$ be an open set of finite Lebesgue measure $|\Omega| < \infty$. It is known that the spectrum of the Dirichlet Laplacian in Ω is discrete (G.Rozenblum, E.H.Lieb).

In 1972 F.Berezin and Li-Yau (1983) proved that for the eigenvalues $\{\lambda_k\}$ and any $\gamma \geq 1$

$$\mathrm{Tr}\,(-\Delta - \lambda)_-^\gamma \leq (2\pi)^{-d} \int_\Omega \int_{\mathbb{R}^d} (|\xi|^2 - \lambda)^\gamma \, d\xi dx = L_{\gamma,d}^{cl} \, |\Omega| \lambda^{\gamma+d/2}.$$

In 1980 V.Ivrii proved that

$$\mathrm{Tr}\,(-\Delta - \lambda)_-^\gamma = L_{\gamma,d}^{cl} \, |\Omega| \lambda^{\gamma+d/2} - \frac{1}{4} \, L_{\gamma,d-1}^{cl} \, |\partial\Omega| \lambda^{\gamma+(d-1)/2}.$$

Recently A.D.Melas, T.Weidl, H. Kovarik, S. Vugalter, and T. Weidl, Il'in have obtained uniform estimate of the remainder term. However, such a remainder was always expressed in terms $|\Omega|$ rather that $|\partial\Omega|$.

Main result.

For $x \in \Omega$ and $\theta \in \mathbb{S}^{d-1}$ denote

$$d(x,\theta) = \inf\{|t| : x + t\theta \notin \Omega,\ t \in \mathbb{R}\}.$$

Theorem.

If $\gamma \geq 3/2$ we have

$$\operatorname{Tr}(-\Delta - \lambda)_-^\gamma \leq L_{\gamma,d}^{cl} \int_\Omega \left(\lambda - \frac{1}{4d(x,\theta)^2}\right)^\gamma dx.$$

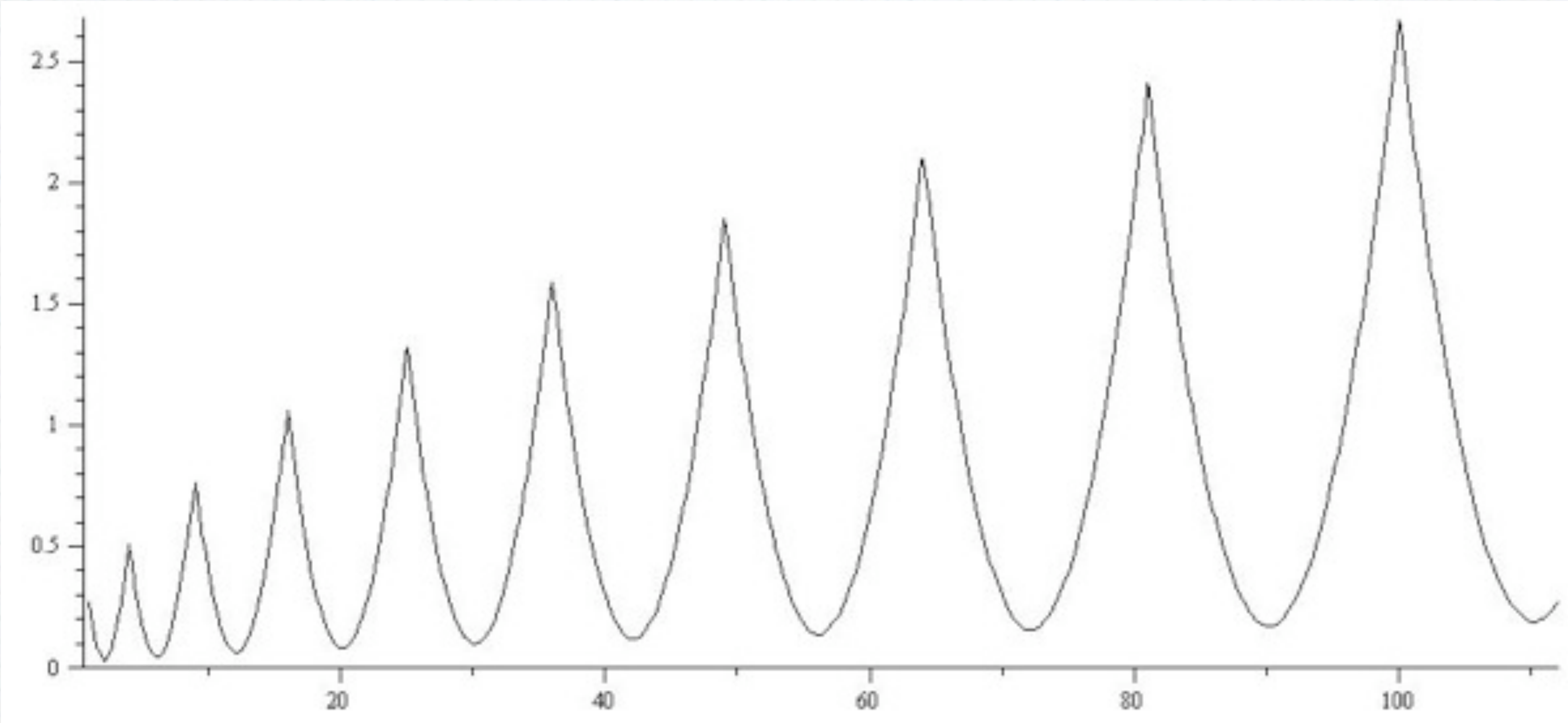
The proof is based on a simple surprising lemma.

Lemma.

Let $\gamma \geq 1$. Then

$$\sum_{k=1}^\infty (\lambda - k^2)_+^\gamma \leq L_{\gamma,1}^{cl} \int_0^\pi \left(\lambda - \frac{1}{4\,\delta(t)^2}\right)_+^{\gamma+1/2} dt,$$

where $\delta(t) := \min\{t, \pi - t\}$. The constants $L_{\gamma,1}^{cl}$ and $1/4$ in this inequality are sharp for any $\gamma \geq 1$ and the inequality fails if $\gamma < 1$.



Here is the graph of the difference

$$f(\lambda) = \sum_{k=1}^{\infty} (\lambda - k^2)_+ - L_{1,1}^{cl} \int_0^{\pi} \Big(\lambda - \frac{1}{4\delta(t)^2}\Big)_+^{3/2} \, dt.$$

Corollary. Let $\Omega \subset \mathbb{R}^d$ be a convex domain with smooth boundary and assume that the curvature of $\partial\Omega$ is bounder by $1/R$. Then for any $\gamma \geq 3/2$ and all $\lambda > 0$ we have

$$\operatorname{Tr}(-\Delta - \lambda)_-^\gamma \;\leq\; L_{\gamma,d}^{cl}\,|\Omega|\,\lambda^{\gamma+d/2} - C_1|\partial\Omega|\lambda^{\gamma+(d-1)/2} + C_2\frac{|\partial\Omega|}{R}\,\lambda^{\gamma+d/2-1},$$

where $C_{1,2} = C(\gamma, d)$.

Proof. The proof is based on integrating over $\theta \in \mathbb{S}^{d-1}$ the inequality

$$\operatorname{Tr}(-\Delta - \lambda)_-^\gamma \leq L_{\gamma,d}^{cl} \int_\Omega \left(\lambda - \frac{1}{4d(x,\theta)^2}\right)^\gamma dx.$$

and Steiner's inequality

$$\left(1 - \frac{d-1}{R}\right)|\partial\Omega| \leq |\,\partial\Omega_t|,$$

where $\Omega_t = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \geq t|$.