Isoperimetric inequalities for Lévy processes: Finite dimensional distributions.[1]

Rodrigo Bañuelos[2]

Department of Mathematics
Purdue University
West Lafayette, IN 47907

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Circular shapes are extremals for many problems under the assumptions of fixed area.

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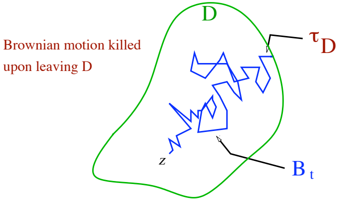
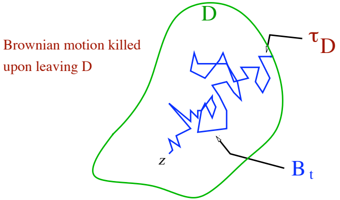
"The isoperimetric theorem, deeply rooted in our experience and intuition so easy to conjecture, but not so easy to prove, is an inexhaustible source of inspiration."

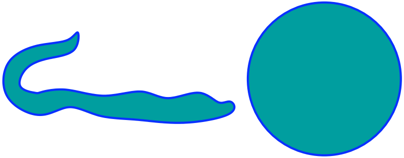
G. Pólya: Mathematics and Plausible Thinking


D
Brownian motion killed
upon leaving D
τ_D
z
B_t



Question

Assuming same volume, which of the following figures has the largest survivable time and where should the "random walker" start to maximized its chances of being alive by time t?



Answer is “obvious": Right hand shape starting at the origin.

“Theorem" (Fixed volume)

The finite dimensional distributions of a large class of Lévy processes, including all subordinations of Brownian motion such relativistic Brownian motion, stable processes, and their relativistic versions, are majorized by those of their symmetrized versions in symmetrized sets.

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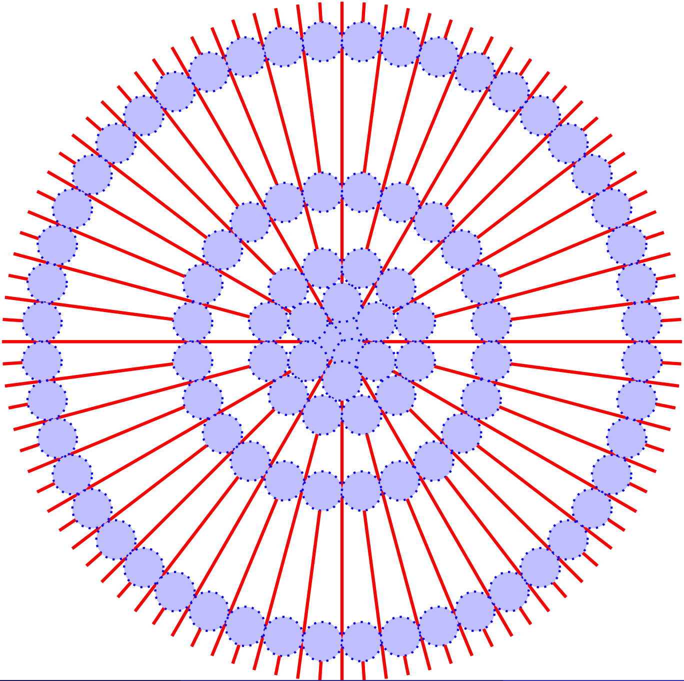
More precisely

Theorem 1

For any Lévy process X_t with Lévy measure absolutely continuous to the Lebegue measure it holds that for any m and any open sets $D_j \subset \mathbb{R}^d$, $1 \leq j \leq m$, $D^* =$ ball same volume

$$P_z\{X_{t_1} \in D_1, \ldots, X_{t_m} \in D_m\} \leq P_0\{X^*_{t_1} \in D^*_1, \ldots, X^*_{t_m} \in D^*_m\}$$

for all times $0 < t_1 < t_2 < \cdots < t_m < \infty$, where X^*_t is a rotationally symmetric process constructed from X_t.



For any D, let $d_D(z)$ be the distance from $z \in D$ to the boundary ∂D. Set

$$r_D = \sup_{z \in D} d_D(z)$$

For a large class of domains D (not all)

$$\lambda_1(D) \approx \frac{1}{r_D^2}$$

In fact for all simply connected domains D in the plane, (R.B.–T. Carroll (1994))

$$\frac{0.6194}{r_D^2} \leq \lambda_1(D) \leq \frac{j_0^2}{r_D^2}$$

Question

Amongst the class of all **simply connected plane domains** *with fixed inradius "which one(s)" maximize "lifetime" or minimize the eigenvalue?*

Theorem 2 (Fixed inradius)

$D \subset \mathbb{R}^d$ iconvex with inradius $r_D < \infty$. For any subordination of Brownian motion X_t and $0 < t_1 < t_2 < \cdots < t_m < \infty$

$$P_z\{X_{t_1} \in D, \ldots, X_{t_m} \in D\} \leq P_0\{X_{t_1} \in S_{r_D}, \ldots, X_{t_m} \in S_{r_D}\},$$

$$S_{r_D} = \mathbb{R}^{d-1} \times (-r_D, r_D) = \text{ infinite strip (slab) of width } 2r_D.$$

1. These give the classical isoperimetric inequality (Dido's property), Pólya-Szegö isoperimetric capacity, Faber-Krahn, heat kernels, Greens functions, trace of semigroups (including Schrödinger),...
2. Of interest here is the case when the generator of the process is not a local operator such as fractional powers of the Laplacian or any "subordinations" of the Brownian motion.
3. Fixed volume "generalized" Heat Kernel isoperimetry inequalities for the Laplacian and elliptic operators in domains of $\mathbb{R}^n$, spheres, hyperbolic space, etc., have been proved by many people: Luttinger, Friedberg-Luttinger, Talenti, Bandle, Brock-Solynin, Morpurgo, Burchard-Schmuckenschläger, ...

Theorem (For Brownian motion)

Amongst all regions D of fixed volume the ball maximizes the lifetime of Brownian motion in the distribution sense. That is, for all D, $t > 0$, $x \in D$,

$$P_x\{\tau_D > t\} \leq P_0\{\tau_{D^*} > t\} \qquad (1)$$

$$\int_D P_x\{\tau_D > t\}dx \leq \int_{D^*} P_x\{\tau_{D^*} > t\}dx \qquad (2)$$

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$$(1) \iff P_0\{\tau_{D^*} \leq t\} \leq P_x\{\tau_D \leq t\} \tag{3}$$

$$(4) \iff \int_{D^*} P_x\{\tau_{D^*} \leq t\}dx \leq \int_D P_x\{\tau_D \leq t\}dx \tag{4}$$

Known

$$\lim_{t \to 0} \frac{1}{\sqrt{t}} \int_D P_x\{\tau_D \leq t\}dx = \frac{2}{\sqrt{\pi}}\sigma(\partial D)$$

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$$\lim_{t\to\infty} \frac{1}{t} \log P_x\{\tau_D > t\} = -\lambda_1(D), \qquad x \in D$$

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2. Also probably first noticed, in the general setting by M. Kac.

$$\lim_{t \to \infty} e^{\lambda_1(D)t} P_x\{\tau_D > t\} = \varphi(x) \int_D \varphi_1(y) dy, \quad x \in D$$

3. φ_1 is the ground state eigenfunctions corresponding to $\lambda_1(D)$. This convergence is uniformly in $x \in D$ for many D's but not all! Follows from “intrinsic-ultracontractivity".

In term of Dirichlet heat kernel $P_t^D(x,y)$ for Laplacian in D

$$P_x\{\tau_D > t\} \leq P_0\{\tau_{D^*} > t\}, \iff$$

$$\int_D P_t^D(x,y)dy \leq \int_{D^*} P_t^D(0,y)dy,$$

and

$$\int_D P_x\{\tau_D > t\}dx \leq \int_{D^*} P_x\{\tau_{D^*} > t\}dx \iff$$

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Remark

Special cases of more general inequalities in C. Bandle's "Isoperimetric inequalities and applications" Ch IV for uniformly elliptic operator with bounded measurable coefficients with ellipticity constant 1. That is,

$$L = \sum_{j,k} \partial_j\left(a_{jk}\partial_k\right), \quad \sum_{j,k} a_{jk}\xi_j\xi_k \ge |\xi|^2.$$

Observed by many including Aizenman and Simon who first wrote it down

$$\begin{aligned} P_x\{\tau_D > t\} &= P_x\{B_s \in D;\ \forall s, 0 < s \leq t\} \\ &= \lim_{m\to\infty} P_x\{B_{jt/m} \in D, j = 1, 2, \dots, m\} \\ &= \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}(x - x_1)\cdots p_{t/m}(x_m - x_{m-1})dx_1 \dots dx_m \end{aligned}$$

$$p_t(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t|\xi|^2} d\xi = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$$

In fact: (Via Brownian bridge or Trotter product formula)

$$p_t^D(x, y) = \lim_{m\to\infty} \int_D \cdots \int_D p^2_{t/m}(x - x_1)\cdots p^2_{t/m}(y - x_{m-1})dx_1 \dots dx_{m-1},$$

Theorem (Luttinger 1973)

Let $f_1, \ldots, f_m$ be nonnegative functions in R^d. For any $x_0 \in D$ we have

$$\int_{D^m} \prod_{j=1}^{m} f_j(x_j - x_{j-1}) dx_1 \cdots dx_m \leq \int_{\{D^*\}^m} f_1^*(x_1) \prod_{j=2}^{m} f_j^*(x_j - x_{j-1}) dx_1 \cdots dx_m.$$

D^*=ball center at zero and and same volume as D

Theorem (Brascamp–Lieb–Luttinger (1975), (1977))

$$\int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} f_j \left(\sum_{i=1}^{k} b_{ji} x_i \right) dx_1 \cdots dx_k \leq \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{m} f_j^* \left(\sum_{i=1}^{k} b_{ji} x_i \right) dx_1 \cdots dx_k,$$

for all positive integers k, m, and any $m \times k$ matrix $B = [b_{ji}]$.

- Roots lie in inequalities of Hardy–Littlewood–Pólya–Riesz

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_1(x_1)H(x_2 - x_1)F_2(x_2)dx_1dx_2 \leq *$$

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Theorem (R. B. Latala, Méndez, 2001 ($d=2$), Méndez 2003, $d \geq 3$)

$D \subset \mathbb{R}^d$ convex inradius $r_D < \infty$, $S = \mathbb{R}^{d-1} \times (-r_D, r_D)$ infinite strip. Let $f_1, \ldots, f_m$ be nonnegative radially symmetric decreasing on $\mathbb{R}^d$. For any $x_0 \in \mathbb{R}^d$,

$$\int_D \cdots \int_D \prod_{j=1}^m f_j(x_j - x_{j-1})\, dx_1 \cdots dx_m \leq$$

$$\int_S \cdots \int_S f_1(x_1) \prod_{j=2}^m f_j(x_j - x_{j-1})\, dx_1 \cdots dx_m.$$

Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: **de Finetti, Kolmogorov, Khintchine, Itô.**

1. Rich stochastic processes, generalizing several basic processes in probability: Brownian motion, Poisson processes, stable processes, subordinators, . . .
2. Regular enough for interesting analysis and applications. Their paths consist of continuous pieces intermingled with jump discontinuities at random times. Probabilistic and analytic properties studied by many.
3. Many Developments in Recent Years:
   - **Applied:** Queueing Theory, Math Finance, Control Theory, Porous Media . . .
   - **Pure:** Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

1. X has independent and stationary increments
2. $X_0 = 0$ (with probability 1)
3. X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(dx) = (2\pi)^{d/2}\widehat{p_t}(\xi)$$

where p_t is the distribution of X_t. Notation (same with measures)

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx, \quad f(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)d\xi$$

The characteristic function has the form $\varphi_t(\xi) = e^{-t\rho(\xi)}$, where

$$\rho(\xi) = -ib \cdot \xi + \langle \mathbb{A} \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{\{|x|<1\}}(x)\right) \nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $n \times n$ matrix $\mathbb{A}$ and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right) \nu(dx) < \infty.$$

$\rho(\xi)$ is called the **symbol** of the process or the **characteristic exponent.** The triple $(b, \mathbb{A}, \nu)$ is called the **characteristics of the process.**

Converse also true. Given such a triplet we can construct a Lévy process.

1. **Standard Brownian motion:**
   With $(0, I, 0)$, I the identity matrix,
   $$X_t = B_t, \qquad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**
   $(0, \mathbb{A}, 0)$, X_t is "generalized" Brownian motion, mean zero, covariance
   $$E(X_s^j X_t^i) = a_{ij} \min(s, t)$$
   X_t has the normal distribution (assume here that $det(\mathbb{A}) > 0$)
   $$\frac{1}{(2\pi t)^{d/2}\sqrt{\det(\mathbb{A})}} \exp\left(-\frac{1}{2t} x \cdot \mathbb{A}^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, \mathbb{A}, 0)$ get gaussian processes with drift:
   $$X_t = bt + G_t$$

4. **Poisson Process:** Poisson Process $X_t = \pi_t(\gamma)$ of intensity $\gamma > 0$ is a Lévy process with $(0, 0, \gamma\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{\pi_t(\gamma) = m\} = \frac{e^{-\gamma t}(\gamma t)^m}{m!}, \quad m = 1, 2, \ldots$$

π_t continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : \pi_t(\gamma) = m\}$$

5. **Compound Poisson Process** Let $Y_1, Y_2, \ldots$ be i.i.d. and independent of the π_t with distribution ν.

$$X_t = Y_1 + Y_2 + \cdots + Y_{\pi_t(\gamma)} = S_{\pi_t(\gamma)}$$

$$\begin{aligned} E[e^{i\xi\cdot X_t}] &= \sum_{m=0}^{\infty} P\{\pi_t = m\}\, E[e^{i\xi\cdot S_m}] \\ &= \sum_{m=0}^{\infty} \frac{e^{-\gamma t}(\gamma t)^m}{m!} \left(\widehat{\nu}(\xi)\right)^m = e^{-\gamma t(1-\widehat{\nu}(\xi))} \\ \Rightarrow &\quad \rho(\xi) = \gamma \int_{\mathbb{R}^d} \left(1 - e^{ix\cdot\xi}\right) \nu(dx) \end{aligned}$$

6. **Relativistic Brownian motion** According to quantum mechanics, a particle of mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2 c^4 + c^2 |p|^2} - mc^2$$

where c is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

7. **The rotationally invariant stable processes:** These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

$\alpha = 2$ is **Brownian motion.** $\alpha = 1$ is the **Cauchy processes.** Transition probabilities:

$$P_x\{X_t^\alpha \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad \text{any Borel} \qquad A \subset \mathbb{R}^d$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t|\xi|^\alpha} d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad \alpha = 2, \qquad \text{Brownian motion}$$

$$p_t^1(x) = \frac{C_d\, t}{(|x|^2 + t^2)^{\frac{d+1}{2}}}, \qquad \alpha = 1, \qquad \text{Cauchy Process}$$

For any $a > 0$, the two processes

$$\{X_{(at)}\,;\ t \geq 0\} \quad \text{and} \quad \{a^{1/\alpha} X_t\,;\ t \geq 0\},$$

have the same finite dimensional distributions **(self-similarity).**

In the same way, the transition probabilities scale similarly to those for BM:

$$p_t^{\alpha}(x) = t^{-d/\alpha} p_1^{\alpha}(t^{-1/\alpha} x)$$

8. Subordinators

A subordinator is a one-dimensional Lévy process $\{T_t\}$ such that (i) $T_t \geq 0$ a.s. for each $t > 0$ and (ii)] $T_{t_1} \leq T_{t_2}$ a.s. whenever $t_1 \leq t_2$

Theorem (Laplace transform characterization)

$$E(e^{-\lambda T_t}) = e^{-t\psi(\lambda)},\ \lambda > 0,$$

$$\psi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds)$$

$b \geq 0$ and the Lévy measure satisfies $\nu(-\infty, 0) = 0$ and $\int_0^\infty \min(s, 1)\nu(ds) < \infty$. ψ is called the Laplace exponent of the subordinator.

Example ($\alpha/2$–Stable subordinator): $\psi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$ gives the with $b = 0$ and

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1 - \alpha/2)}\, s^{-1-\alpha/2}\, ds$$

Example 2 (Relativistic stable subordinator): $0 < \alpha < 2$ and $m > 0$, $\Psi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} e^{-m^{2/\alpha}s} s^{-1-\alpha/2} ds$$

Example 3 (Gamma subordinator): $\Psi(\lambda) = \log(1+\lambda)$.

$$\nu(ds) = \frac{e^{-s}}{s} ds$$

Many others: "Geometric stable subordinators, iterated geometric stable subordinators, Bessel subordinators,. . . "

Theorem

If X is an arbitrary Lévy process and T is a subordinator independent of X, then $Z_t = X_{T_t}$ is a Lévy process.

$$p_{Z_t}(A) = \int_0^\infty p_{X_s}(A) p_{T_t}(ds)$$

- If $X_t =$ Brownian motion, Z_t is called subordinate Brownian motion.
- $\alpha/2$ subordinator gives the α-rotationally invariant stable process and $p_t^\alpha(x-y) = \int_0^\infty p_s^2(x-y) g_{\alpha/2}(t,s) ds, \quad 0 < \alpha < 2.$

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{t\rho(\xi)} \widehat{f}(\xi) d\xi$$

with generator

$$\begin{aligned} Af(x) &= \frac{\partial T_t f(x)}{\partial t}\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \rho(\xi) \hat{f}(\xi) d\xi \end{aligned}$$

A pseudo diff operator, in general

From the Lévy–Khintchine formula (and properties of the Fourier transform),

$$Af(x) = \sum_{i=1} b_i \partial_i f(x) \quad + \quad \sum_{i,j} a_{i,j} \partial_i \partial_j f(x)$$
$$+ \quad \int \Big[f(x+y) - f(x) - y \cdot \nabla f(x) \chi_{\{|y|<1\}} \Big] \nu(dy)$$

1. Standard Brownian motion (running at twice the usual speed):
$$Af(x) = \Delta f(x)$$
2. Poisson Process of intensity γ:
$$Af(x) = \gamma \Big[f(x+1) - f(x) \Big]$$
3. Rotationally Invariant Stable Processes of order $0 < \alpha < 2$, **Fractional Diffusions:**
$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &= A_{\alpha,d} \int \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy \end{aligned}$$

Lemma

Suppose ν is absolutely continuous with respect to Lebesgue measure with density $\phi(x)$. Then $\phi^*(x)\,dx$ is also a Lévy measure.

Set $\mathbb{A}^* = (\det \mathbb{A})^{1/d}\, I_d$ and define X_t^* to be the (rotationally invariant) Lévy process in $\mathbb{R}^d$ associated to the triple $(0, \mathbb{A}^*, \phi^*(x)dx)$.

$$\begin{aligned} \rho^*(\xi) &= \langle \mathbb{A}^* \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[1 - e^{i\,\xi\cdot x} \right] \phi^*(x)\,dx \\ &= \langle \mathbb{A}^* \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} [\,1 - \cos(\xi \cdot x)\,]\, \phi^*(x)\,dx, \end{aligned}$$

using the fact that ϕ^* is symmetric and $y \to \sin(\xi \cdot x)$ is antisymmetric.

Theorem

$$E_z\left[\prod_{i=1}^{m} f_i(X_{t_i})\right] \le E_0\left[\prod_{i=1}^{m} f_i^*(X_{t_i}^*)\right],$$

for all $0 \le t_1 \le \ldots \le t_m$.

Remark (Outline of proof)

The building blocks for Lévy processes are compound Poisson processes and Gaussian processes. Compound Poisson are random walks ran up to a Poisson process. The following is a key lemma.

Remark

We refer to Bañuelos and P. Méndez-Hernández, JFA 2010, for details and careful statements of all results below. Here we only illustrate, largely abusing precision and rigor.

Lemma ($S_n = X_1 + X_2 + \cdots + X_n$, X_i iid $\approx \phi(x)dx$)

$k_1 \leq \ldots \leq k_m$ *nonnegative integers.*

$$E\left[\prod_{i=1}^{m} f_i(x_0 + S_{k_i})\right] \leq E\left[\prod_{i=1}^{m} f_i^*(S_{k_i}^*)\right],$$

Same as

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left[\prod_{i=1}^{m} f_i\left(\sum_{j=0}^{k_i} x_j\right)\right] \prod_{i=1}^{k_m} \phi(x_i)\, dx_1 \ldots dx_{k_m}$$

$$\leq \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left[\prod_{i=1}^{m} f_i^*\left(\sum_{j=1}^{k_i} x_j\right)\right] \prod_{i=1}^{k_m} \phi^*(x_i)\, dx_1 \ldots dx_{k_m}$$

$$E^x\left[\prod_{i=1}^{m} f_i(S_{\pi_{t_i}})\right] = \sum_{k_1 \leq k_2 \leq \ldots \leq k_m}^{\infty} P\left[\pi_{t_1} = k_1, \ldots, \pi_{t_m} = k_m\right] E\left[\prod_{i=1}^{m} f_i(x + S_{k_i})\right.$$

Consequences

$$\tau_D^X = \inf\{t > 0 : X_t \notin D\}$$

1. If ψ is a nonnegative increasing function, then
$$E^z\left[\psi\left(\tau_D^X\right)\right] \le E^0\left[\psi\left(\tau_{D^*}^{X^*}\right)\right],$$
for all $z \in D$. In particular for all $0 < p < \infty$.
$$E^z\left[\left(\tau_D^X\right)^p\right] \le E^0\left[\left(\tau_{D^*}^{X^*}\right)^p\right].$$

2. For all $z \in D$, $t > 0$ and nonnegative Borel functions f,
$$\int_D f(w)\, p_D^X(t,z,w)\, dw \le \int_{D^*} f^*(w)\, p_{D^*}^{X^*}(t,0,w)\, dw, \qquad (5)$$
$p_D^X(t,z,w)$ “heat kernel" for killed “heat" semigroup
$$T_t f(x) = E_x\left[f(X_t); \tau_D^X > t\right]$$

3. If X_t and X_t^* are transient, then

$$\int_D f(w) G_D^X(z,w)\,dw \le \int_{D^*} f^*(w) G_{D^*}^{X^*}(0,w)\,dw, \tag{6}$$

$G_D^X(z,w)$, $G_{D^*}^{X^*}(0,w)$ Green's functions for X_t, X_t^*, respectively.

4. By (5), (6) and Alvino-Trombetti-Lions (1989) for all increasing convex functions $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\int_D \Psi\left(p_D^X(t,z,w) \right) dw \le \int_{D^*} \Psi\left(p_{D^*}^{X^*}(t,0,w) \right) dw,$$

and

$$\int_D \Psi\left(G_D^X(z,w) \right) dw \le \int_{D^*} \Psi\left(G_{D^*}^{X^*}(w,0) \right) dw,$$

- (For elliptic operators these hold for all nonnegative increasing functions, See C. Bandle, page 214.)
- Similar inequalities for trace (including "Schródinger perturbations"), heat content, torsional rigidity, Faber-Krahn, ...

For Subordination of BM

The Finite dimensional distributions of subordinate Brownian motion are integrals of the finite dimensional distributions of Brownian motion.

Thus for α-stable, for example

$$P_x\{X^\alpha_{t_1} \in D_1, \ldots, X^\alpha_{t_m} \in D_m\}$$
$$= \int_0^\infty \ldots \int_0^\infty P_x\{B_{s_1} \in D_1, B_{(s_1+s_2)} \in D_2, \ldots, B_{(s_1+s_2+\cdots+s_n)} \in D_m\}$$
$$\times \prod_{i=1}^m g_{\alpha/2}(t_i - t_{i-1}, s_i)\, ds_1 \ldots ds_m.$$

Theorem (Subordination of Brownian motion, convex, fixed inradius)

If $X_t = B_{T_t}$ subordinate Brownian motion then the equalities hold for convex domains D of fixed inradius r_D with D^* replaced by strip.

Theorem (Subordination of Brownian motion, convex, fixed inradius)

If $X_t = B_{T_t}$ subordinate Brownian motion then the equalities hold for convex domains D of fixed inradius r_D with D^* replaced by strip. In particular, for stable order $0 < \alpha \le 2$ for all $x \in D$ and all $t > 0$,

$I_{r_D} = (-r_D, r_D)\,, \quad S_{r_D} = \mathbb{R}^{d-1} \times (-r_D, r_D)$

$$P_x\{\,\tau_D^\alpha > t\,\} \le P_0\{\,\tau_{S_{r_D}}^\alpha > t\,\} = P_0\{\,\tau_{I_{r_D}}^\alpha > t\,\}$$

Corollary (Fractional Laplacian in convex domains of fixed inradius)

$\lambda_{D,\alpha}$ the first Dirichlet eigenvalue for $(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$ in D:

$$\lambda_{I_{r_D},\alpha} \le \lambda_{D,\alpha}$$

- for $\alpha = 2$, the eigenvalue inequality is well known (Hersh, Protter, . . .)
- For $\alpha = 2$, the exit time distribution proved by R.B, Latała, Méndez-Hernández($d = 2$) and R.B. Kröger ($d > 2$) and for all $0 < \alpha \le 2$ and all d by Méndez-Hernández.
- For $\alpha = 2$ and $d = 2$, Green function inequalities were proved by R.B-T. Carroll-E. Housworth.

Problem

Determine conditions for equality.

Remark

For stable processes the case of equality has been studied by Porpurgo in his Duke 2002 paper; (see also Burchard and Schmuckenschläger 2003 for Brownian motion). A general "Lieb formula" is also studied by Morpurgo. Such formula probably holds here but this has not being written down yet.

Conjecture

The inequalities

$$\int_D \Psi\big(\, p_D^X(t,z,w)\,\big)\,dw \leq \int_{D^*} \Psi\big(\, p_{D^*}^{X^*}(t,0,w)\,\big)\,dw,$$

and

$$\int_D \Psi\big(\, G_D^X(z,w)\,\big)\,dw \leq \int_{D^*} \Psi\big(\, G_{D^*}^{X^*}(w,0)\big)\,dw,$$

should hold for all increasing functions $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ and not just for convex increasing. Open even for stable processes!