ISOPERIMETRIC INEQUALITIES ON NONPOSITIVELY CURVED SPACES

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Abstract. These are notes from the intensive course given during the conference "From Carthage to the world : the isoperimetric problem of Queen Dido and its mathematical consequences" held in Carthage in May 2010.

The course intends in giving an overview of what is known about isoperimetric inequalities on nonpositively curved spaces, more precisely on Cartan-Hadamard manifolds. All the course turns around a conjecture, which is still open, which asserts that the Euclidean isoperimetric inequality should hold on complete, simply-connected Riemannian manifolds of nonpositive sectional curvature (Cartan-Hadamard manifolds). The aim is to give some backgrounds to understand why this conjecture should be true and to understand the outlines of the proofs of some of the main results on the subject, leaving details to the reader who can refer to the original papers.

First, I will give a proof of the isoperimetric inequality in the Euclidean case, which is the simplest Cartan-Hadamard manifold. I also suggest some alternative proofs and give the links between isoperimetric and Sobolev inequalities. In a second part, I will quickly recall some facts about Riemannian geometry which are useful to convince oneself that the conjecture should be true. In a third part, I state the conjecture concerning Cartan-Hadamard manifolds and I review the three cases where the answer to the conjecture is known to be positive, namely in dimensions 2, 3 and 4. What is fun and strange is that the proofs have nothing to do one with the other and that they work only for one specific dimension. At last[1], I will give some local results and briefly explain how to prove optimal local isoperimetric inequalities passing through optimal Sobolev inequalities.

1. Isoperimetric and Sobolev inequalities in $\mathbb{R}^n$

1.1. **Isoperimetric inequalities in the plane.** It was known since ancient times that, if you take a domain in $\mathbb{R}^2$, then

$$L^2 \geq 4\pi A\ ,$$

the case of equality being achieved only by discs. For a rigorous proof, one had to wait for the nineteenth century[2]. If we consider domains in $\mathbb{R}^2$ bounded by simple Jordan curves, it's sufficient to consider convex domains. Indeed, passing to the convex hull clearly decreases the perimeter and increases the area. This is really specific to the plane and considerably simplifies the proof of the 2-d isoperimetric inequalities compared to the proof of its generalization to higher dimensions.

Let me give a simple proof of this inequality, which gives moreover some of the Bonnesen inequalities (see [5]) and so gives easily the case of equality. Let Ω be a convex domain in $\mathbb{R}^2$. Let R be the radius of the smallest circumbscribed circle to Ω. We let then $\Omega(R)$ the

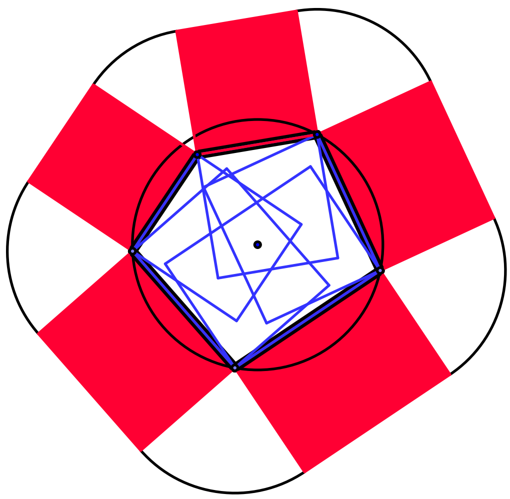
[1]This was not done during the course.

[2]For a short history of the problem and proofs of the isoperimetric inequality in $\mathbb{R}^2$, see Burago-Zalgaller [6]).

set of points in $\mathbb{R}^2$ which are at a distance less than or equal to R from Ω. Let $A(R)$ be the area of this domain $\Omega(R)$. Then we have that

$$A(R) = A + LR + \pi R^2 \leq 2LR\,.$$

In order to prove this, the easiest way is to prove the inequality for convex polygons and to pass to the limit. For convex polygons, the proof clearly follows from the following picture :



The area of $\Omega(R)$ is equal to the area of Ω plus the sum of the area of the red rectangles (which gives LR) plus the sum of the area of the angular sectors (which gives πR^2). On the other hand, looking at the area of the rectangles outside (red) and inside (blue[3]) the polygon, their total area is $2LR$. Compared to the area of $\Omega(R)$, we forget outside all the angular sectors but we count them twice inside so it compensates and we count at least once any point inside the convex polygons since any point is at a distance less than R from the boundary. This proves the above inequalities. Now this gives

$$A \leq LR - \pi R^2 = -\pi\left(R - \frac{L}{2\pi}\right)^2 + \frac{L^2}{4\pi} \leq \frac{L^2}{4\pi}$$

with equality if and only if $L = 2\pi R$, which clearly implies that Ω is a disc.

1.2. **Isoperimetric inequalities in higher dimensions.** We have seen that proving the isoperimetric inequality in the plane was not really difficult and could be achieved with elementary geometry arguments. A natural generalization of the isoperimetric property of the circle in higher dimensions consists in saying that for any domain Ω in $\mathbb{R}^n$, the volume of the boundary of Ω is greater than or equal to the volume of the sphere of radius R, where R is such that the ball of radius R has the same volume than Ω. Taking into account the scaling properties of the volume, this is equivalent to saying that, for any smooth domain $\Omega \subset \mathbb{R}^n$,

$$\frac{Vol(\partial\Omega)}{Vol(\Omega)^{\frac{n-1}{n}}} \geq \frac{Vol\,(\partial B_1)}{Vol(B_1)^{\frac{n-1}{n}}} = K_n^{-1}\,.$$

Note that

$$K_n^{-1} = \omega_{n-1}^{\frac{1}{n}} n^{\frac{n-1}{n}}$$

where $\omega_{n-1} = Vol\,(\partial B_1)$.

[3]which are the reflections of the red ones inside the polygon.

Proving this inequality is much harder than proving the 2-d isoperimetric inequatity. Let me give a proof due to Gromov[4]. Since one perfectly knows what the isoperimetric domains are in $\mathbb{R}^n$, namely the balls, a natural way to prove the isoperimetric inequality is to find a map from a domain Ω into the ball of same volume which preserves the volume and decreases the area[5]. The proof of Gromov I will describe below uses the Knothe map. There is another nice proof in [9] in the same spirit using the Brenier map coming from optimal transport.

Let's go with the proof of Gromov. Let Ω be a smooth domain of $\mathbb{R}^n$. By scaling invariance of the isoperimetric inequality, we can assume that the volume of Ω is the volume of the unit ball. Let us construct a map from Ω into B in the following way : for $x \in \Omega$, let $\tilde{x}_1$ be such that

$$Vol\left(\{y_1 > x_1\} \cap \Omega\right) = Vol\left(B \cap \{y_1 > \widetilde{x}_1\}\right) .$$

Let us set $f(x)_1 = \widetilde{x}_1$.

Let $\widetilde{x}_2$ be s.t.

$$Vol\left(\{y_1 > x_1\} \cap \Omega \cap \{y_2 > x_2\}\right) = Vol\left(B \cap \{y_1 > \widetilde{x}_1\} \cap \{y > \widetilde{x}_2\}\right)$$

and let us set $f(x)_2 = \widetilde{x}_2$. And so on by induction.

This map is continuous, regular in Ω. There are some problems of regularity on the boundary of Ω (in the parts which are concave) but let's ignore them. We have at least enough regularity to apply Stokes theorem : seeing f as a vector field in Ω, we can write that

$$\int_\Omega div(f)dx = \int_{\partial\Omega} f\overrightarrow{\nu}\, d\sigma \leq |\partial\Omega| \|f\|_\infty = |\partial\Omega| .$$

By the construction of f, one readily checks that f is volume-preserving and $f(x)_i$ does depend only on $x_1, \dots, x_i$. Thus the matrix $\partial_i f_j$ is of the form

$$(\partial_i f_j(x)) = \begin{pmatrix} \lambda_1(x) & * & * & * & * \\ 0 & ... & * & * & * \\ 0 & 0 & \lambda_i(x) & * & * \\ 0 & 0 & 0 & ... & * \\ 0 & 0 & 0 & 0 & \lambda_n(x) \end{pmatrix}$$

with $\prod_{i=1}^n \lambda_i(x) = 1$. Using the arithmetic-geometric inequality, we can write that

$$div(f)(x) = \sum_{i=1}^n \lambda_i(x) \geq n \left(\prod_{i=1}^n \lambda_i(x)\right)^{\frac{1}{n}} = n .$$

Thus we obtain that

$$nVol_g\left(\Omega\right) \leq Vol_g\left(\partial\Omega\right)$$

which is exactly the isoperimetric inequality since $Vol_g\left(\Omega\right) = Vol_g\left(B_1\right)$ and $nVol_g\left(B_1\right) = Vol_g\left(\partial B_1\right)$.

One can also trace back the case of equality by noticing first that $f = \nu$ on the boundary and that $\lambda_i \equiv 1$ for all i. Writing f after some translation as

$$f(x) = (x_1, x_2 + a(x_1), x_3 + b(x_1, x_2), \dots)$$

[4]I did not find any exact reference for this proof. M. Berger, in one of his book on differential geometry, writes that this proof is due to Gromov.

[5]This is a kind of symetrization process. There are in fact many ways to realize such a map.

and using the fact that the boundary of Ω is characterized by the fact that $|f|^2 = 1$, one can deduce that Ω is necessarily a ball. There is work to be done here but we will not do it in these notes.

Thus we have a proof of the isoperimetric inequality in $\mathbb{R}^n$ (up to the fact that we did not finish the proof of the fact that only balls are achieving the case of equality). Anyway, this fact can be deduced in another way, as we shall see in the next subsection.

1.3. **Properties of extremal domains.** Let us get some properties of isoperimetric domains, which are also true in any Riemannian manifold. It's just easier to obtain these properties in the Euclidean space. This will give us another approach to prove the isoperimetric inequality in the Euclidean space, which reveals to be much harder than the way Gromov used but it's interesting in view of possible generalisations to curved spaces. We shall see some applications of it in section 3.2.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let us denote by Σ its boundary. Let us denote by $d\mu_\Sigma$ the volume element on Σ induced from the Euclidean volume element. We want to understand how this volume element varies when the surface flows in the direction of its exterior normal.

We let ν be the unit normal vector of Σ pointing outward[6]. This normal unit vector may be seen as a map from Σ into S^{n-1}, the unit sphere of $\mathbb{R}^n$. Its differential may then be seen as a self-adjoint linear map from the tangent space of Σ at x into the tangent space of the sphere at $\nu(x)$. The fact that it is self-adjoint is not obvious but one can find a proof of it in any classical textbook which deals with hypersurfaces in $\mathbb{R}^n$. This differential has $n-1$ eigenvalues, $(\lambda_i)_{1\le i\le n-1}$, and, by definition, the mean curvature of Σ at the point x is

$$H(x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\lambda_i(x)\ .$$

Since the hypersurface Σ may be seen in a neighbourhood of any of its point as a graph over its tangent space, and since the computations we want to do are purely local, we shall assume in the following that

$$\Sigma = \{(x, f(x))\,,\ x\in U\}$$

locally, where U is a neighbourhood of 0 in $\mathbb{R}^{n-1}$, f is a smooth real-valued function in U with $f(0)=0$ and $\nabla f(0)=0$. Let us remark that $0\in\Sigma$ (which can always be done thanks to a translation) and that the tangent space at 0 is $\mathbb{R}^{n-1}\times\{0\}$ (which can also always be assumed after rotation). The unit normal vector is given by

$$\nu(x) = \frac{1}{\sqrt{1+|\nabla f|^2}}\left(\partial_i f, -1\right)\ .$$

In particular, the mean curvature at a point $(x, f(x))$ is

$$H(x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\partial_i\left(\frac{\partial_i f}{\sqrt{1+|\nabla f|^2}}\right)$$

which is the so-called mean curvature operator (for graphs). At 0, this gives

$$H(0) = -\frac{1}{n-1}\Delta f(0)\ .$$

[6]In general, there are two choices of normal but, here, since Σ is the boundary of a domain, one can make a choice, depending on where Ω is with respect to its boundary.

Let us now look at

$$\Sigma_t = \{(x, f(x)) + t\nu(x),\ x \in U\}$$

for t small. The volume element $d\mu_{\Sigma_t}$ in the chart U is given by

$$d\mu_{\Sigma_t}(x,t) = \sqrt{det g_t(x)}\, dx$$

where

$$(g_t(x))_{ij} = (e_i(x,t), e_j(x,t))$$

with

$$e_i(x,t) = \left(\partial_i\left(x_j + t\nu_j\right), \partial_i \nu_n\right)\ .$$

We can then write that

$$\frac{d}{dt} d\mu_{\Sigma_t}(0,0) = \frac{1}{2}\sqrt{det g_0(0)}\, tr\left(g_0(0)^{-1}\frac{d}{dt}g_t(0,0)\right) dx = \frac{1}{2} tr\left(g_0(0)^{-1}\frac{d}{dt}g_t(0,0)\right) d\mu_{\Sigma_0}(0)\ .$$

Noting that $g_0(0) = Id$, the computations are easy to carry out and one simply finds that

$$\frac{d}{dt} d\mu_{\Sigma_t}(0,0) = (n-1)H(0) d\mu_{\Sigma_0}(0)\ .$$

Thus we have proved the following :

Proposition 1. *If Σ is a smooth hypersurface of dimension $n-1$ in $\mathbb{R}^n$ and ν is a unit normal vector, and if $\Sigma_t = \Sigma + tf(x)\nu(x)$ for some function f defined on Σ, then*

$$\frac{d}{dt} Vol\left(\Sigma_t\right)(t=0) = (n-1)\int_\Sigma f(x)H(x)\, d\mu_\Sigma(x)$$

where $H(x)$ is the mean curvature of Σ at x.

Then it is easy to check that, if Ω_0 is an isoperimetric domain, that is

$$\frac{Vol\left(\partial\Omega_0\right)}{Vol\left(\Omega_0\right)^{\frac{n-1}{n}}} = \inf_{Vol(\Omega)=Vol(\Omega_0)} \frac{Vol\left(\partial\Omega\right)}{Vol\left(\Omega\right)^{\frac{n-1}{n}}}\ ,$$

its boundary $\Sigma_0 = \partial\Omega_0$ has constant mean curvature H_0 given by

$$H_0 = \frac{1}{n}\frac{Vol\left(\partial\Omega_0\right)}{Vol\left(\Omega_0\right)}\ .$$

Thus the boundaries of isoperimetric domains have constant mean curvature.

This gives another approach to prove the isoperimetric inequality :

- Prove that there exists some isoperimetric domain.

- Classify (or get informations) on the domains which have constant mean curvature boundaries.

As I said, it's not the most efficient way to prove the isoperimetric inequality in the Euclidean space. The first point in particular leads to serious difficulties ans we can not, in particular, ensure that these isoperimetric domains have smooth boundaries (in high dimensions). The second step was achieved by Alexandrov [1] : the only domains having CMC boundaries are balls. Anyway, this approach is natural and can be useful also in Riemannian manifolds (see sections 3.2 and 4.2).

1.4. **Isoperimetric and Sobolev inequalities.** Before leaving the Euclidean world, let us have a look at the link between isoperimetric and Sobolev inequalities. This will permit to see a proof of the rearrangement inequalities, which are so useful in the isoperimetric world.

Let $1 \leq p < n$. There exists $C_{n,p} > 0$ s.t

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\, dx\right)^{\frac{n-p}{np}} \leq C_{n,p} \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\right)^{\frac{1}{p}} \tag{I_1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. I shall give a quick proof of this fact.

Lemma 1. *If $C_{n,1}$ exists, then $C_{n,p}$ exists for all $p \geq 1$.*

Proof - Apply (I_1) to $|u|^{\frac{p(n-1)}{n-p}}$ to obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\, dx\right)^{\frac{n-1}{n}} &\leq C_{n,1}\left(\int_{\mathbb{R}^n} \left|\nabla |u|^{\frac{p(n-1)}{n-p}}\, dx\right|\right) \\ &= C_{n,1}\frac{p(n-1)}{n-p}\int_{\mathbb{R}^n} |u|^{\frac{n(p-1)}{n-p}} |\nabla u|\, dx \\ &\leq C_{n,1}\frac{p(n-1)}{n-p}\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}}\, dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\right)^{\frac{1}{p}}\end{aligned}$$

which implies (I_p) with $C_{n,p} = C_{n,1}\frac{p(n-1)}{n-p}$. ◊

Lemma 2. *Inequality (I_1) holds for all $u \in C_c^\infty(\mathbb{R}^n)$ with $C(n,1) = \frac{1}{2}$.*

Proof - For $u \in C_c^\infty(\mathbb{R}^n)$, we write that

$$u(x) = \int_{-\infty}^{x_i} \partial_i u = -\int_{x_i}^{\infty} \partial_i u$$

so that

$$|u(x)| \leq \frac{1}{2}\int_{-\infty}^{\infty} |\partial_i u(....,t,x_{i+1}....)|\, dx$$

which leads to

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(\prod_{i=1}^n \int_{-\infty}^{\infty} |\partial_i u(....,t,x_{i+1}....)|\, dx\right)^{\frac{1}{n-1}}.$$

And so we have that

$$\|u(x)\|_{\frac{n}{n-1}} \leq \frac{1}{2}\left(\int_{\mathbb{R}^n}\left(\prod_{i=1}^n \int_{-\infty}^{\infty} |\partial_i u(....,t,x_{i+1}....)|\, dx\right)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}$$

One has just to prove by induction on n (and it's an easy exercise) that

$$\left(\int_{\mathbb{R}^n}\left(\prod_{i=1}^n \int_{-\infty}^{\infty} F_i(x_1,...,x_{i-1},t,x_{i+1},...,x_n)\, dx\right)^{\frac{1}{n-1}}\right)^{n-1} \leq \prod_{i=1}^n \int_{\mathbb{R}^n} F_i(x)\, dx$$

to conclude.

We have obtained Sobolev inequalities in the Euclidean space. They are for from being optimal. What about the optimal inequalities ? In fact, we have the following proposition, due to Federer and Fleming [12] :

Proposition 2. *The H_1^1-inequality is equivalent to the isoperimetric inequality. In other words,*

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \leq K\int_{\mathbb{R}^n} |\nabla u|\,dx \Longleftrightarrow |\Omega|^{\frac{n-1}{n}} \leq K|\partial\Omega|$$

Proof - Take $\Omega \subset \mathbb{R}^n$ a smooth domain. Define u_ε by

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega,\ d\,(x,\partial\Omega) \geq \varepsilon \\ \dfrac{d\,(x,\partial\Omega)}{\varepsilon} & \text{if } x \in \Omega,\ d\,(x,\partial\Omega) \leq \varepsilon \\ 0 & \text{if } x \notin \Omega \end{cases}$$

It's not in $C_c^\infty(\Omega)$ but its gradient is in L^1 and it can be approximated by smooth functions. Then

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|\,dx \longrightarrow |\partial\Omega|$$

and

$$\int_{\mathbb{R}^n} |u_\varepsilon|^{\frac{n}{n-1}}\,dx \longrightarrow |\Omega|$$

as $\varepsilon \longrightarrow 0$. This proves $\Longrightarrow$.

In order to prove that the isoperimetric inequality implies the H_1^1-Sobolev inequality, we need the co-area formula, see lemma 3 below. Let $u \in C_c^\infty(\mathbb{R}^n)$. We have that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx = \frac{n}{n-1}\int_0^{+\infty} t^{\frac{1}{n-1}}V(t)\,dt$$

where $V(t)$ is the volume of

$$\Omega_t = \{x \in \mathbb{R}^n \text{ s.t. } |u(x)| > t\}$$

by Fubini's theorem. Applying lemma 3 below with $\Phi = 1$, we get that

$$\int_{\mathbb{R}^n} |\nabla u|\,dx = \int_0^{+\infty} A(t)\,dt$$

where $A(t) = |\partial\Omega_t|$. Using the isoperimetric inequality, we can write that

$$K\int_{\mathbb{R}^n} |\nabla u|\,dx \geq \int_0^{+\infty} V(t)^{\frac{n-1}{n}}\,dt \geq \left(\frac{n}{n-1}\int_0^{+\infty} t^{\frac{1}{n-1}}V(t)\,dt\right)^{\frac{n-1}{n}} = \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}.$$

The second inequality is true for any non-increasing function $V(t)$. It's a simple exercise and is left to the reader. ♢

In the above proof, we have used the following :

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$, $\bar{\Omega}$ compact, $f : \bar{\Omega} \mapsto \mathbb{R}$ in $C^0\left(\bar{\Omega}\right) \cap C^\infty(\Omega)$ s.t. $f = 0$ on $\partial\Omega$. We let $\Gamma(t) = |f|^{-1}(t)$. Then we have that*

$$dV_{\Gamma(t)} = \frac{dA_t\,dt}{|\nabla f|}$$

for all regular value t of f where dA_t is the surface measure on $\Gamma(t)$ induced by the Euclidean one. In particular, we have that

$$\int_\Omega \Phi \left|\nabla f\right| \, dx = \int_0^{+\infty} \left(\int_{\Gamma(t)} \Phi \, dA_t \right) dt$$

for all functions $\Phi \in L^1(\Omega)$.

For a proof, we refer for instance to Chavel [8].

Let's turn now to optimal Sobolev inequalities for $p > 1$. Let $u \in C_c^\infty(\mathbb{R}^n)$. We may approximate it by Morse functions. We set

$$\Omega_t = \{x \in \mathbb{R}^n \text{ s.t. } |u(x)| > t\} \text{ and } \Gamma_t = \partial\Omega_t$$

and we let $V(t)$ and $A(t)$ be their respective volumes. We let $u^\star$ be a radial non-increasing function defined by

$$V^\star(t) = Vol\left(\{x \in \mathbb{R}^n \text{ s.t. } |u^\star(x)| > t\}\right) = V(t) \, .$$

It is clear that $\|u^\star\|_q = \|u\|_q$ for all $q \geq 1$. Now we write that

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx = \int_{\mathbb{R}^n} |\nabla u|^{p-1} |\nabla u| \, dx = \int_0^{+\infty} \left(\int_{\Gamma_t} |\nabla u|^{p-1} \, dA_t \right) dt$$

thanks to lemma 3. Then, by Hölder's inequalities,

$$\int_{\Gamma_t} |\nabla u|^{p-1} \, dA_t \geq A(t)^p \left(\int_{\Gamma_t} |\nabla u|^{-1} \, dA_t \right)^{1-p} = A(t)^p \left(-V'(t)\right)^{1-p}$$

where equality holds if and only if $|\nabla u|$ is constant on Γ_t for a.e. t (which is the case for $u^\star$). Since $A(t) \geq A^\star(t)$ (with equality for $u^\star$) thanks to the isoperimetric inequality, we obtain that

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx \, .$$

Thus the symmetric rearrangement decreases the L^p-norms of the gradient. As a consequence, it's sufficient to obtain a Sobolev inequality for radial non-increasing functions to extend it to all functions. This was first remarked by Aubin [2] and Talenti [17]. The Sobolev inequality for radial functions was obtained by Bliss [4] in 1930.

This we have that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{np}} \leq K(n,p) \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

for all functions u with gradient in $L^p(\mathbb{R}^n)$, equality being achieved by

$$u(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}$$

and all $\mu u(\lambda x + x_0)$, and only by these functions.

We have seen that :

1) the isoperimetric inequality is equivalent to the H_1^1-Sobolev inequality.

2) the isoperimetric inequality gives optimal Sobolev inequalities of all order $1 < p < n$ (up to solving the radial case).

This remains in some sense true on Riemannian manifolds because the co-area formula still holds on Riemannian manifolds. These remarks give another idea to attack isoperimetric problems : get optimal Sobolev inequalities for all $p > 1$ (it's easier than for $p = 1$) and pass to the limit. We shall see an example of this approach in section 4.3.

2. Riemannian manifolds of nonpositive curvature

In this section, we just recall some well-known facts in Riemannian geometry which permits to get an intuition why the Euclidean isoperimetric inequality should hold on simply-connected complete Riemannian manifolds of nonpositive sectional curvature. We refer the reader to any textbook in Riemannian geometry for these results.

The exponential map at a point x in a complete Riemannian manifold M is a map from $\mathbb{R}^n$ into M which associates to any vector $v \in \mathbb{R}^n$ the point in M at which the geodesic starting from x with speed v at time 0 arrive at time 1, that is

$$\exp_x(v) = \gamma_{(x,v)}(1) .$$

We need here to identify the tangent space of M at x with $\mathbb{R}^n$, which can be done by choosing some orthonormal basis for the tangent space[7].

The main result we shall need is the following : if we let

$$D\exp_x(tu)(tv) = X(t) \tag{1}$$

for two vectors u and v, then X satisfies the following equation :

$$\left(\ddot{X}, Y\right) + \mathcal{R}\left(\gamma', X, \gamma', Y\right) = 0 \tag{2}$$

for any vector field Y along the geodesic $\exp_x(tu)$ with initial data $X(0) = 0$ and $\dot{X}(0) = v$. Here the dot stands for the covariant derivative of a vector field along the geodesic and $\mathcal{R}$ is the Riemann curvature. Such a vector field X is called a Jacobi field.

The Riemann curvature measures the defect of parallelism of small rectangles. But more important to us is the sectional curvature which associates to any two-plane in the tangent space T_xM with an orthonormal basis (X, Y) the real number $\mathcal{R}(x)\,(X, Y, X, Y)$. The sectional curvature measures the infinitesimal speed at which two geodesics starting from x in the direction of the two-plane spanned by their initial speed go away one from the other. They go away one from the other faster than in the Euclidean case if and only if $K_g(X, Y) \leq 0$. One just has to think of the geodesics on the standard sphere and on the hyperbolic space to get a clear picture of this. Locally, the behaviour of the geodesics is given by the sectional curvature and follows what happens either in the sphere, or in the Euclidean space or in the hyperbolic space.

More important to us are the following consequences of equations (1) and (2) :

Theorem 1 (Cartan-Hadamard). *Let (M, g) be a smooth complete simply-connected Riemannian manifold of nonpositive sectional curvature. Then the exponential map at any point x is a smooth diffeomorphism from $\mathbb{R}^n$ into M. Such a manifold is called a Cartan-Hadamard manifold.*

The idea of the proof is that the exponential map can become singular if and only if there exists a nonzero vector field satisfying the equation (2) which vanishes at $t = 0$ and after some time $t \neq 0$. But, in nonpositive sectional curavture, it is impossible since

$$\frac{d}{dt^2}|X|^2 = 2|\dot{X}|^2 + 2\left(\ddot{X}, X\right) = 2|\dot{X}|^2 - \mathcal{R}\left(\gamma', X, \gamma', X\right) \geq 0$$

[7]Thus the exponential map is defined up to this choice.

so that vanishing at two different times implies vanishing in the whole interval. Then one has to use the simply-connectedness to conclude.

Another important consequence based on equations (1) and (2) is the Rauch comparison theorem which says that, in the exponential chart at x, if the sectional curvatures along the geodesic $\exp_x(tu)$ are less than K, then the volume element in the direction of u grows faster than in the model space of constant sectional curvature K_0. Once again, we refer to any textbook in Riemannian geometry for this result.

3. The isoperimetric conjecture on Cartan-Hadamard manifolds in dimensions 2, 3 and 4

Let (M, g) be a smooth complete simply-connected Riemannian manifold of dimension n with nonpositive sectional curvature. Note that you can think of M as $\mathbb{R}^n$ since they are diffeomorphic. The isoperimetric conjecture is the following[8] :

Conjecture : For any smooth domain in M,

$$Vol_g(\partial\Omega) \geq Vol_\xi(\partial B_r)$$

where $r > 0$ is such that $Vol_g(\Omega) = Vol_\xi(B_r)$.

Note that you can make the same conjecture if you replace nonpositive sectional curvature by $K_g \leq -k$ and if you compare with balls in the hyperbolic space with curvature $-k$. Indeed, we know that, in hyperbolic spaces, isoperimetric domains are balls.

This conjecture is rather natural if you think to the two interpretations of the sectional curvature we gave (behaviour of geodesics and growth of the volume element). It's worthy to note that, for geodesic balls, the inequality holds thanks to the comparison theorem of Rauch recalled in the previous section. However, in general Cartan-Hadamard manifolds, isoperimetric domains will not be balls. This conjecture seems clearly to be true but it seems also to be really difficult to prove.

As far as we know, here are the situations where we know something :

- It is true on hyperbolic spaces.

- It is true in dimensions 2, 3 and 4 on any Cartan-Hadamard manifold.

- It is true for domains of large volume in Cartan-Hadamard manifolds with sectional curvature $K_g \leq -k < 0$.

- It is true for domains of small diameters[9] if the scalar curvature of the manifold is negative.

In the rest of these notes, I would like to give ideas of the proofs of these different results, except for the hyperbolic space where one can take almost any Euclidean proof and transpose it.

3.1. Dimension 2.

This is the first dimension in which the conjecture was proved and this is this result which certainly gave birth to the conjecture. It is due to Weil [18] in 1926. Beckenbach and Rado [3] gave an independent proof in 1933, capitalizing on a result of Carleman [7] of 1921.

Let $(\mathbb{R}^2, g)$ with $K_g \leq 0$ and g complete.

[8] It's hard to say who stated this conjecture first since it's rather natural to ask this question since the results of Weil in 2d.

[9] and will soon be extended to domains of small volumes under some mild additional assumptions on the geometry at infinity.

Lemma 4. *It is sufficient to prove the conjecture for connected domains.*

Proof - Note that this remark is true in all dimensions but let us give a proof in dimension 2, which one can easily adapt.

If $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 disconnected, then

$$A = A_1 + A_2 \text{ and } L = L_1 + L_2$$

with obvious notations and

$$L^2 - 4\pi A \geq L_1^2 - 4\pi A_1 + L_2^2 - 4\pi A_2 + 2L_1 L_2$$

which clearly proves that, if $L^2 - 4\pi A \leq 0$, then there exists i such that $L_i^2 - 4\pi A_i < 0$. It clearly permits to prove that it is sufficient to consider connected domains. ♢

Lemma 5. *It is sufficient to prove it for topological discs.*

Proof - This is true only in dimension 2 (it is not known in higher dimensions). Let $\Omega \subset \mathbb{R}^2$. If the boundary has many connected components, take the exterior one and fill it to form a new domain which has less boundary and more area than Ω. ♢

Let Ω be topological disk. It can be conformally represented by the unit disc so that the problem reduces to proving the isoperimetric inequality on the disc with a metric conformal to the Euclidean one, $g = e^u \xi$, with nonpositive sectional curvature, which means $\Delta u \leq 0$. Thus we need to prove that for any smooth function on the disc with $\Delta u \leq 0$,

$$\left(\int_{\partial D} e^u \, d\sigma \right)^2 \geq 4\pi \int_D e^{2u} \, dx \, .$$

Let v be such that $\Delta v = 0$ in D and $v = u$ on ∂D. Then $v \geq u$ and if we are able to prove the above inequality for v, then u will also satisfy it. Let $v^\star$ be the harmonic conjugate of v so that $e^{v+iv^\star}$ is holomorphic. Then, since it is nowhere 0, it can be written as $e^{v+iv^\star} = \varphi^2$ with φ holomorphic. And we need to prove that

$$\left(\int_{\partial D} |\varphi|^2 \, d\sigma \right)^2 \geq 4\pi \int_D |\varphi|^4 \, dx \, .$$

It can easily be done writing

$$\varphi = \sum_{n=0}^{+\infty} a_n z^n \, .$$

Indeed, we have[10]

$$\int_{\partial D} |\varphi|^2 \, d\sigma = 2\pi \sum_{n=0}^{+\infty} |a_n|^2$$

and

$$\int_D |\varphi|^4 \, dx = \pi \sum_{n=0}^{+\infty} \frac{|b_n|^2}{n+1}$$

avec $b_n = \sum_{k+l=n} a_l a_k$. The desired inequality follows from Cauchy-Schwarz. ♢

This conformal representation is often useful in 2d but it can clearly not be extended to higher dimensions.

[10]To be really rigorous, one needs to do all this argument on smaller discs of radius $1 - \varepsilon$ and to pass to the limit at the end.

3.2. **Dimension** 3. Let (M,g) be any Riemannian manifold. Define the isoperimetric profile of M by

$$I_M(V) = \inf_{\Omega\subset M,\,|\Omega|_g=V} |\partial\Omega|_g$$

for $0 \le V \le Vol_g(M)$. Proving the isoperimetric conjecture amounts to proving that

$$I_M(V) \ge I_0(V) = \omega_{n-1}^{\frac{1}{n}} n^{\frac{n-1}{n}} V^{\frac{n-1}{n}}$$

for any Cartan-Hadamard manifold of dimension n where I_0 is the isoperimetric profile of the Euclidean space.

Note that

$$I_0'(V) = (n-1)\omega_{n-1}^{\frac{1}{n-1}} I_0(V)^{-\frac{1}{n-1}} .$$

In general, we do not know if the isoperimetric profile of a Riemannian manifold is differentiable but it is continuous and at least admits a left-derivative that we shall denote by D_-I_M. If one can prove that

$$D_-I_M(V) I_M(V)^{\frac{1}{n-1}} \ge (n-1)\omega_{n-1}^{\frac{1}{n-1}} = I_0'(V) I_0(V)^{\frac{1}{n-1}}$$

for all $V > 0$ on a Cartan-Hadamrd manifold, then the conjecture would be proved by integrating. That's the way Kleiner attacked the problem in [13].

Let us assume for the moment that $I_M(V)$ is achieved and let Ω_V be a smooth domain of volume V such that

$$I_M(V) = |\partial\Omega_V| .$$

Take a smooth family Ω_t of domains such that $\Omega_0 = \Omega_V$ and look at the graph $(|\Omega_t|, |\partial\Omega_t|)$. At $t=0$, it has derivative $(n-1)H_V$, the mean curvature of $\partial\Omega_V$, which must be constant (see section 1.3 for this discussion). But the graph $(|\Omega_t|, |\partial\Omega_t|)$ lies above $I_M(V)$ so that

$$D_-I_M(V) \ge (n-1)H_V .$$

Thus the goal is to prove that the mean curvature of $\partial\Omega_V$, H_V, satisfies that

$$H_V |\partial\Omega_V|^{\frac{1}{n-1}} \ge \omega_{n-1}^{\frac{1}{n-1}} .$$

Note that this inequality is scale-invariant and that the right-hand side is the value of the left-hand side for some Euclidean sphere in $\mathbb{R}^{n+1}$.

Let us now assume that $n=3$. Kleiner then proved that any boundary of a domain with constant mean curvature satisfy the above inequality. In other words, if $\Omega \subset M$, and if $\partial\Omega = \Sigma$ has constant mean curvature H, then

$$H Vol_g(\Sigma)^{\frac{1}{2}} \ge \sqrt{4\pi} .$$

Let us assume to simplify (considerably) that Σ is a topological sphere (or Ω a topological ball[11]). Then, by the Gauss-Bonnet theorem, we know that

$$\int_\Sigma S_{\tilde g}\, dv_{\tilde g} = 4\pi$$

where $\tilde g$ is the induced metric on Σ and $S_{\tilde g}$ is its scalar curvature. By the Gauss formula, we also know that

$$S_{\tilde g}(x) = 2K_g(T_x\Sigma) + 2\lambda_1\lambda_2$$

[11] which is rather natural for some isoperimetric domain in a Cartan-Hadamard manifold but we do not know how to prove this.

where λ_1 and λ_2 are the eigenvalues of $D\nu$. But

$$K_g\left(T_x\Sigma\right) \leq 0$$

since (M,g) has nonpositive sectional curvature and

$$2\lambda_1\lambda_2 = \frac{1}{2}\left(\lambda_1+\lambda_2\right)^2 - \frac{1}{2}\left(\lambda_1-\lambda_2\right)^2 \leq 2H^2 .$$

Plugging this information in the Gauss-Bonnet formula, we directly obtain that

$$H^2 Vol_g\left(\partial\Sigma\right) \geq 4\pi$$

which is the result we were looking for.

Of course, unlike Kleiner, I cheated a lot. First, it's not clear at all that isoperimetric domains do exist (it is even clear that in some cases, they do not exist). In order to deal with this, Kleiner consider the isoperimetric profile in a ball where isoperimetric domains do exist. However, they are not anymore smooth, even if we are in dimension 3. In fact, they are smooth inside the ball and $C^{1,1}$ at the boundary by a result of White [19]. It is sufficient to have a weak notion of mean curvature with which one can work as above. The second problem is that isoperimetric domains are not necessarily topological balls. We refer the reader to the paper of Kleiner to see how he reduces, by a clever argument passing through the convex hull of the domain, the general inequality on the mean curvature to the case where Ω is a topological ball.

The approach of Kleiner is certainly the most natural one. It also permitted to Morgan and Johnson [14] to prove a local version of the conjecture (see section 4.2). However, proving the inequality on the mean curvature of the boundary of isoperimetric domains is much harder (is it even true ?) in higher dimensions because the Gauss-Bonnet formula is less nice.

3.3. **Dimension** 4. In 1984, Croke proved in [10] an inequality of the form

$$Vol_g\left(\partial\Omega\right) \geq C_n Vol_g\left(\Omega\right)^{\frac{n-1}{n}}$$

for any domain Ω of any Cartan-Hadamard manifold of dimension n. Thus there is a universal isoperimetric inequality, depending only one the dimension, on Cartan-Hadamard manifolds. Quite miraculously, the constant Croke obtains is optimal in dimension 4, and not really good, compared to the conjectured optimal one, in higher dimensions. The proof of Croke relies on tools of integral geometry, more precisely Santalo's formula. For a book on integral geometry, see Santalo [16].

Let $M \subset \mathbb{R}^n$, and g be a metric in $\mathbb{R}^n$ with nonpositive sectional curvature. We let UM be the unitary tangent bundle, that is

$$UM = \left\{u = \left(x, \vec{u}\right),\, x \in M,\, \vec{v} \in T_xM,\, |\vec{v}|_g = 1\right\} .$$

We let $U^+\partial M$ be the subset of UM consisting of points $x \in \partial M$ together with a unit tangent vector which satisfies $\left(\nu, \vec{u}\right)_g > 0$ where ν is the unit interior normal vector to ∂M at x. For any $u \in U^+\partial M$, we let $l(u)$ be the exit time of the geodesic starting at x with speed $\vec{u}$. In other words, if $(x,\vec{u}) \in U^+\partial M$, we consider the geodesic γ_u which satisfies $\gamma_u(0) = x$ and $\gamma_u'(0) = \vec{u}$. This geodesic belongs to M for a certain amount of time. We let $l(u)$ be the first time such that it hits the boundary. This will always happen since we are in a space of nonpositive sectional curvature.

Consider now

$$\Phi : U^+\partial M \times [0, l(u)] \mapsto UM$$

defined by

$$\Phi(u,t) = \gamma_u(t)\ .$$

It is clearly one to one. Indeed, it is surjective since, given any $v \in UM$, you have to look at the point in $U^+\partial M$ where the geodesic starting from x at speed $-\vec{v}$ hits the boundary to get a preimage of v. It is injective since two geodesics starting from a different point in $U^+\partial M$ can not arrive at the same point UM, by uniqueness of a geodesic starting from one point with a given speed. It is a diffeomorphism almost everywhere. On UM, there is a canonical measure given by g. Let $d\mu_{UM}$ denote its volume element. It's not difficult to check that

$$\Phi^\star d\mu_{UM} = \cos u d\mu_{U^+\partial M} dt$$

where $\cos u = (\vec{u}, \nu)_g$. Indeed the geodesic flow is preserving $d\mu_{UM}$ so that one just has to look at what happens at time $t = 0$. And to pass from $d\mu_{UM}$ to $d\mu_{U^\partial M} dt$, one just has to change ν by u, which gives the desired formula.

As a consequence, we have Santalo's formula :

$$\int_{UM} f(v)dv = \int_{U^+\partial M} \left(\int_0^{l(u)} f\left(\gamma_u(t)\right) \cos u dt \right) du\ . \tag{3}$$

For $f \equiv 1$, this leads to

$$\omega_{n-1} Vol_g\left(M\right) = \int_{U^+\partial M} l(u) \cos u du\ . \tag{4}$$

Let us set $ant(u) = -\gamma_u'\left(l(u)\right)$. This map from $U^+\partial M$ into itself is measure preserving and we have that[12]

$$\int_{U^+\partial M} g(u) \cos u\, du = \int_{U^\partial M} g\left(antu\right) \cos u\, du\ . \tag{5}$$

Let us use Hölder's inequalities to get that

$$\omega_{n-1} Vol_g\left(M\right) \le \left(\int_{U^\partial M} \frac{l(u)^{n-1}}{\cos(antu)}\, du \right)^{\frac{1}{n-1}} \left(\int_{U^+\partial M} \cos\left(antu\right)^{\frac{1}{n-2}} \left(\cos u\right)^{\frac{n-1}{n-2}}\ du \right)^{\frac{n-2}{n-1}} \tag{6}$$

with equality if and only if

$$\lambda(u) = K \cos(antu)^{\frac{1}{n-2}} \cos(u)^{\frac{1}{n-2}}\ . \tag{7}$$

Using once again Hölder's inequalities, we can write that

$$\begin{aligned}
&\int_{U^+\partial M} \cos\left(antu\right)^{\frac{1}{n-2}} \left(\cos u\right)^{\frac{n-1}{n-2}}\ du \\
&\le \left(\int_{U^+\partial M} \cos\left(antu\right)^{\frac{2}{n-2}} \cos u\, du \right)^{\frac{1}{2}} \left(\int_{U^+\partial M} \left(\cos u\right)^{\frac{n}{n-2}}\ du \right)^{\frac{1}{2}} \\
&= \int_{U^\partial M} \left(\cos u\right)^{\frac{n}{n-2}}\ du \\
&= Vol_g\left(\partial M\right) \omega_{n-2} \int_0^{\frac{\pi}{2}} \left(\cos t\right)^{\frac{n}{n-2}} \left(\sin t\right)^{n-2}\ dt := \alpha_n Vol_g\left(\partial M\right)
\end{aligned}$$

where we used (5) to get the third line. Note that we have equality if and only if

$$\cos(u) = K \cos(antu)$$

[12] see Croke [10] for the details.

but K has to be 1 since both $\cos u$ and $\cos(antu)$ are in $(0,1]$. So we have equality if and only if

$$\cos(u) = \cos(antu) \ . \tag{8}$$

Coming back to (6), we have obtained so far that

$$\omega_{n-1} Vol_g(M) \le \alpha_n^{\frac{n-2}{n-1}} Vol_g(\partial M)^{\frac{n-2}{n-1}} \left(\int_{U^+\partial M} \frac{l(u)^{n-1}}{\cos(antu)} du \right)^{\frac{1}{n-1}} \ . \tag{9}$$

Let us work now in the exponential chart at $x \in \partial M$. Writing that

$$dv_g(\exp_x(t\vec{u})) = F(u,r) du dr \ ,$$

we have that

$$\int_{U_x^+\partial M} \frac{F(u,l(u))}{\cos(antu)} du = Vol_g(A_x) \le Vol_g(\partial M)$$

where $A_x = \Phi(\{(u,l(u)), u \in U_x^+\partial M\})$. And we have equality if the entire boundary can be seen from the point x, that is $A_x = \partial M$. Now, by the Rauch comparison theorem, we know that $F(u,l(u)) \ge l(u)^{n-1}$ with equality if and only if all the sectional curvatures along the geodesic γ_u are 0. Combining all these results, we can write that

$$\int_{U^\partial M} \frac{l(u)^{n-1}}{\cos(antu)} du \le Vol_g(\partial M)^2$$

with equality if and only if M is flat and convex. Coming back to (9), we thus obtain that

$$Vol_g(\partial M) \ge C_n Vol_g(M)^{\frac{n-1}{n}}$$

with

$$C_n = \omega_{n-1}^{\frac{n-1}{n}} \alpha_n^{-\frac{n-2}{n}}$$

where

$$\alpha_n = \omega_{n-2} \int_0^{\frac{\pi}{2}} (\cos t)^{\frac{n}{n-2}} (\sin t)^{n-2} \, dt \ .$$

This is valid in all dimensions.

In order to see whether the constant is sharp or not, one can compute it or one can trace back the case of equality. There is equality in all what we did if the domain M is flat, convex, if $\cos(u) = \cos(antu)$, which means that the domain is a ball in $\mathbb{R}^n$ and if $l(u) = K\cos(u)^{\frac{2}{n-2}}$, which is possible if and only if this ball is in $\mathbb{R}^4$. Thus the inequality is sharp in dimension 4, equality being achieved only for flat balls. In higher dimensions, the constant is not so good, compared to what we are waiting for. But, nevertheless, even in higher dimensions, it is interesting to get some constant which depends only on the dimension of the Cartan-Hadamard manifold you consider.

4. Local optimal isoperimetric inequalities on manifolds

4.1. **The result on large volumes of Yau.** Assume (M,g) is a simply connected Riemannian manifold of dimension $n \ge 2$ with sectional curvature $K_g \le -k < 0$. Then we have that

$$Vol_g(\partial\Omega) \ge (n-1)\sqrt{k} Vol_g(\Omega)$$

for all smooth domains in M. This result, due to Yau [20] in 1975, gives a linear isoperimetric inequality in this situation. In particuler, we see that the isoperimetric conjecture holds for domains of large volumes if the sectional curvature is bounded from above by a negative constant.

The proof of such a result is rather easy and comes directly from the Rauch comparison theorem. Indeed, let us work in the exponential chart at some point x_0 and let us denote by $r(x)$ the distance from x_0 to x. Then, we have that r is smooth on $M \setminus \{x\}$ since we are on a Cartan-Hadamard manifold and, by Stokes theorem,

$$- \int_\Omega \Delta_g r(x)\, dv_g = \int_{\partial\Omega} \partial_\nu r(x)\, d\sigma_g \leq Vol_g\left(\partial\Omega\right) .$$

Let us remark that in the exponential chart at x, we have that

$$-\Delta_g r(x) = \frac{n-1}{r(x)} + \frac{x^i \partial_i \sqrt{|g|}}{\sqrt{|g|}} \frac{1}{r(x)} \geq -\Delta_{g_{-k}} r(x)$$

where g_{-k} is the standard metric in the hyperbolic space of curvature $-k$. Now you clearly have equality for a ball in the hyperbolic space of curvature $-k$ so that we easily get the above result without computations. Otherwise, one has to check, doing the computation with this metric g_{-k}, that it works.

4.2. **A first local result on compact manifolds.** Let's turn now to results for small domains on manifolds. The first result I would like to mention is the following, due to Morgan and Johnson [14] in 2000 :

Theorem 2. *Let (M, g) be a smooth compact Riemannian manifold of dimension n with sectional curvature $K_g < K_0$. Then there exists $V_0 > 0$ such that*

$$I_M(V) \geq I_{K_0}(V)$$

for all $0 < V < V_0$.

This result permits to say that, in compact manifolds of sectional curvature less than K_0, the isoperimetric inequality of the corresponding model space holds for domains of small volumes.

The idea of the proof is roughly the following. Take a sequence of isoperimetric domains Ω_V of volume $V \to 0$. These isoperimetric domains do exist since we are in a compact manifold. They are smooth up to dimension 7 and not a priori smooth for higher dimensions but one can control the singularities. The mean curvature of their boundary is constant. It is not difficult to show that these isoperimetric domains shrink around some point on the manifold as the volume $V \to 0$ and that they look like small balls asymptotically. The reason is that, when they are sufficiently concentrated around a point, the Riemannian metric is so closed from the Euclidean one that isoperimetric domains should be almost balls. This argument can be made precise through a rescaling of the domains to make them of size 1.

Then, since one knows the asymptotic behaviour of the second fundamental form, one can make use of the Gauss-Bonnet formula in higher dimensions and control everything in the same spirit as in the proof of Kleiner. This gives, being really careful, the result. Please see the paper by Johnson and Morgan for more details.

4.3. **An optimal local result.** The result of the previous subsection is far from being optimal. Indeed, as soon as one wants a local result, that's the scalar curvature which should play a role, not the sectional curvature. And, in fact, we were able in [11], to prove the following optimal version of the above theorem by getting an asymptotic expansion of the isoperimetric profile of any manifold up to order 2 for small volumes. Here is the result :

Theorem 3. *Let (M, g) be a Riemannian manifold of dimension n and let $x \in M$. Assume that $S_g(x) < n(n-1)K_0$. Then there exists $\delta_x > 0$ such that for any domain $\Omega \subset B_x(\delta_x)$, we have that*

$$Vol_g(\partial\Omega) \geq Vol_{g_{K_0}}(\partial B_R)$$

where B_R is a ball in the model space of constant sectional curvature K_0 which has the same volume than Ω.

Moreover, if the Riemannian manifold is compact and satisfies $S_g < n(n-1)K_0$ for some $K_0 > 0$, then there exists $V_0 > 0$ such that

$$I(V) \geq I_{K_0}(V)$$

for all $0 \leq V \leq V_0$.

Note that a consequence of this theorem is a local version of the isoperimetric conjecture in Cartan-Hadamard manifolds. Domains of small diameter around a point of negative scalar curvature satisfy the Euclidean isoperimetric inequality. Note that, contrary to theorem 2, we can deal with domains of small diameter in any Riemannian manifold because our proof does not make use of the existence of a "smooth" minimizer. The proof goes through optimal Sobolev inequalities. In fact, we prove optimal H_1^p-Sobolev inequalities for functions compactly supported in small balls for all $p > 1$. Then we let $p \to 1$ to get the isoperimetric inequality.

Our theorem gives an asymptotic expansion of the isoperimetric profile of a compact manifold for small volumes. We have that

$$I(V) = C(n,1)^{-1} V^{\frac{n-1}{n}} \left(1 - c_n \max_M S_g V^{\frac{2}{n}} + o\left(V^{\frac{2}{n}}\right)\right)$$

where c_n is some explicit dimensional constant. This was refined a little bit later by Nardulli [15]. Getting such an asymptotic expansion on complete manifolds under mild assumptions on the geometry at infinity is a work in progress with Stefano Nardulli.

Note at last that our theorem also gives that the isoperimetric domains for small volumes on a Riemannian compact manifold do concentrate at a point where the scalar curvature achieves its maximum.

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